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2025-2026

PG TRB

G.O No. 219 New Syllabus

MATHEMATICS

UNIT-I
Algebra

Your Success is Our Goal...

ALGEBRA

PG TRB (2025-2026)

UNIT-1

FIRST EDITION

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Contents

UNIT I: ALGEBRA

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Chapter 1

Group Theory

1.1 Groups and Some Examples

***Example* 1.1.1:** This is an motivating example for the definition of groups. Let us consider $(\mathbb{Z}, +)$, where $\mathbb{Z}$ is the set of all integers, i.e.,

$$\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\},$$

$+$ is a binary operation on $\mathbb{Z}$, i.e.,

$$+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \text{ defined by } + : (a, b) \mapsto a + b.$$

Observations in this example.

1. Let a, b be any two integers. Then $a + b$ is clearly an integer.

   This property is called *closure*. If a set with a binary operation has only this property, then the set is called a **magma** or a **quasi group**.

2. Let a, b, c be three integers. Then what is $a + b + c$?

   We know how to operate two integers but here the task is to add three integers. We can define $a + b + c$ as

   - $a + b + c = (a + b) + c$ (or)

- $a + b + c = a + (b + c)$.

This property is called *associativity*. If a set with a binary operation has upto this property, that is, satisfies closure and associative, then the set is called a **semi group**.

If $(a + b) + c \neq a + (b + c)$, then we cannot define addition on 3 elements. But in $\mathbb{Z}$, $(a + b) + c = a + (b + c)$.

3. There is a magic number in $\mathbb{Z}$ say 0, because

$$0 + a = a + 0 = a$$

for all $a \in \mathbb{Z}$.

This property is called *existence of identity*, and 0 is called an *identity element*. If a set with a binary operation has upto this property, that is, satisfies closure, associative and existence of identity, then the set is called a **monoid**.

4. For every element $a \in \mathbb{Z}\backslash\{0\}$, there is an element $-a \in \mathbb{Z}\backslash\{0\}$ such that

$$a + (-a) = (-a) + a = 0.$$

For example, for $5 \in \mathbb{Z}$, there exists $-5 \in \mathbb{Z}$ such that $5 + (-5) = 0$ and similarly for $-8 \in \mathbb{Z}$, there exists $8 \in \mathbb{Z}$ such that $-8 + 8 = 0$.

This property is called *existence of inverse*. If a set with a binary operation has upto this property, that is, satisfies closure, associative, existence of identity and existence of inverse, then the set is called a **group**.

5. In $\mathbb{Z}$, we see that $3 + 7 = 7 + 3 = 10$. In general, $a + b = b + a$ for all $a, b \in \mathbb{Z}$.

This property is called *commutativity*. If a set with a binary operation has upto this property, that is, satisfies closure, associative, existence of identity, existence of inverse and commutativity, then the set is called an **abelian group** or a **commutative group**.

Group

Definition 1.1.1 (Group)**.** A non-empty set G together with a binary operation $\star$, i.e.,

$$\star : G \times G \to G$$

is said to be a group, if

1. $(G, \star)$ satisfies closure property, i.e.,

$$\text{for every } a, b \in G \Rightarrow a \star b \in G.$$

2. $(G, \star)$ satisfies associative property, i.e.,

$$\text{for every } a, b \in G, (a \star b) \star c = a \star (b \star c).$$

3. There exists an element $e \in G$ such that

$$a \star e = e \star a = a$$

for all $a \in G$. This e is called an identity element.

4. For every $a \in G$, there exists $a^{-1} \in G$ such that

$$a \star a^{-1} = a^{-1} \star a = e.$$

This a^{-1} is called an inverse of a and vice versa.

***Note* 1:** $(G, \star)$ is called a group or we can say G is a group under $\star$.

Definition 1.1.2 (Abelian group)**.** If in a group $(G, \star)$, if for every $a, b \in G$,

$$a \star b = b \star a$$

is satisfied, then $(G, \star)$ is called an abelian group or a commutative group.

***Example* 1.1.2:** Prove that $(\mathbb{Z}, +)$ is an abelian group.

1. For any integers a & b, $a+b$ is an integer.

2. Clearly, $(\mathbb{Z},+)$ has associativity property.

3. $0 \in \mathbb{Z}$ (Identity)

4. For every $a \in \mathbb{Z}$, there exists $-a \in \mathbb{Z}$ such that $a+(-a)=(-a)+a=0$.

5. For every $a,b \in \mathbb{Z}$, $a+b=b+a$.

***Example* 1.1.3:** Let $\mathbb{Q}=\{\frac{a}{b} \mid a,b \in \mathbb{Z} \quad \& \quad b \neq 0\}$. Prove that $(\mathbb{Q},+)$ is also an abelian group.

1. Let $p,q \in \mathbb{Q}$, $p=\frac{a_1}{b_1}$, $q=\frac{a_2}{b_2}$. Then

$$p+q=\frac{a_1}{b_1}+\frac{a_2}{b_2}=\frac{a_1b_2+a_2b_1}{b_1b_2} \in \mathbb{Q}.$$

2. For $p,q,r \in \mathbb{Q}$, $p=\frac{a_1}{b_1}$, $q=\frac{a_2}{b_2}$, $r=\frac{a_3}{b_3}$.

$$(p+q)+r=\left(\frac{a_1}{b_1}+\frac{a_2}{b_2}\right)+\frac{a_3}{b_3}=\left(\frac{a_1b_2+a_2b_1}{b_1b_2}\right)+\frac{a_3}{b_3}=\frac{a_1b_2b_3+a_2b_1b_3+a_3b_1b_2}{b_1b_2b_3}; \tag{1.1}$$

$$p+(q+r)=\frac{a_1}{b_1}+\left(\frac{a_2}{b_2}+\frac{a_3}{b_3}\right)=\frac{a_1}{b_1}+\left(\frac{a_2b_3+a_3b_2}{b_2b_3}\right)=\frac{b_2b_3a_1+a_2b_1b_3+a_3b_1b_2}{b_1b_2b_3}. \tag{1.2}$$

From (1.1) and (1.2),

$$(p+q)+r=p+(q+r).$$

3. 0 is the identity, since

$$p+0=\frac{a_1}{b_1}+0=0+\frac{a_1}{b_1}=0+p=p.$$

4. For every $p=\frac{a_1}{b_1} \in \mathbb{Q}$, there exists $-\frac{a_1}{b_1} \in \mathbb{Q}$ such that

$$\frac{a_1}{b_1}+\frac{-a_1}{b_1}=\frac{-a_1}{b_1}+\frac{a_1}{b_1}=0.$$

5. For $p, q \in \mathbb{Q}$, we have

$$p + q = \frac{a_1}{b_1} + \frac{a_2}{b_2} = \frac{a_1b_2 + a_2b_1}{b_1b_2} = \frac{a_2b_1 + a_1b_2}{b_1b_2} = \frac{b_1a_2 + b_2a_1}{b_2b_1} = \frac{a_2}{b_2} + \frac{a_1}{b_1} = q + p.$$

   Therefore, $(\mathbb{Q}, +)$ is an abelian group.

***Example* 1.1.4:** The set of all real numbers, $\mathbb{R}$, forms a group under +, i.e., $(\mathbb{R}, +)$ is a group.

The following examples illustrate the properties of a group.

***Example* 1.1.5:** Consider $(\mathbb{Z}, -)$.

1. Let $a, b \in \mathbb{Z}$. Then $a - b \in \mathbb{Z}$ and thus closure property holds.

2. Let $a = 5, b = -6$ and $c = -2$. Then

$$(a - b) - c = (5 - (-6)) - (-2) = (5 + 6) + 2 = 13 \quad \text{and}$$

$$a - (b - c) = 5 - ((-6) - (-2)) = 5 - (-6 + 2) = 5 - (-4) = 5 + 4 = 9.$$

   It is clearly visible that

$$(a - b) - c \neq a - (b - c)$$

   for some $a, b, c \in \mathbb{Z}$. Thus, associativity does not hold in $(\mathbb{Z}, -)$.

Hence, $(\mathbb{Z}, -)$ is a magma or quasi group.

***Example* 1.1.6:** Consider $(\mathbb{Z}, \star)$ where $\star$ is defined as $a \star b = a + b + ab$.

1. Let $a, b \in \mathbb{Z}$. Then $a \star b = a + b + ab \in \mathbb{Z}$. Hence, $(\mathbb{Z}, \star)$ has closure property.

2. Let $a, b, c \in \mathbb{Z}$. Then

$$(a\star b)\star c = (a+b+ab)\star c = a+b+ab+c+(a+b+ab)c = a+b+c+ab+ac+bc+abc \quad \text{and}$$

$$a\star(b\star c) = a\star(b+c+bc) = a+b+c+bc+a(b+c+bc) = a+b+c+bc+ab+ac+abc.$$

   It is clear that $(a \star b) \star c = a \star (b \star c)$ for all $a, b, c \in \mathbb{Z}$.

3. We need to identify if some $e \in \mathbb{Z}$ exists. We expect

$$a \star e = a + e + ae = a \quad \& \quad e \star a = e + a + ea = a.$$

Thus,

$$\begin{aligned} a + e + ae + (-a) &= a + (-a) = 0 \quad \text{since } a \in \mathbb{Z} \Rightarrow -a \in \mathbb{Z}. \\ e + ae &= 0 \\ e(1+a) &= 0 \quad \text{using distributivity in integers} \\ e &= 0 \quad \text{since } 1 + a \neq 0 \text{ for all } a \in \mathbb{Z}. \end{aligned}$$

Thus, $e = 0$ is the required identity with respect to $\star$.

4. For $a \in \mathbb{Z}$, we need to check if there exists $b \in \mathbb{Z}$ such that $a \star b = 0$. Suppose for $a \in \mathbb{Z}$, there exists $b \in \mathbb{Z}$ such that $a \star b = 0$. Then,

$$\begin{aligned} a \star b &= a + b + ab = 0 \\ \Rightarrow b + ab &= -a \\ \Rightarrow b(1+a) &= -a \\ \Rightarrow b &= \frac{-a}{1+a}. \end{aligned}$$

But $\frac{-a}{1+a}$ need not be an integer for every $a \in \mathbb{Z}$. Thus, inverse axiom does not hold.

Hence, $(\mathbb{Z}, \star)$ is a monoid.

***Example* 1.1.7:** Consider $(\mathbb{Q}, \star)$, where $\star$ is defined as $a \star b = a + b + ab$.

Closure, associative and identity axiom holds clearly from the previous example. Let us verify inverse axiom now. Suppose for $a \in \mathbb{Q}$, there exists $b \in \mathbb{Q}$ such that $a \star b = 0$. Then,

$$\begin{aligned} a \star b &= a + b + ab = 0 \\ \Rightarrow b + ab &= -a \\ \Rightarrow b(1+a) &= -a \\ \Rightarrow b &= \frac{-a}{1+a}. \end{aligned}$$

This b is not defined for $a = -1$. Thus, inverse does not exist when $a = -1$. Hence $(\mathbb{Q}, \star)$ does not form a group but it is a monoid.

Let us now restrict the definition of $\star$ to $\mathbb{Q} - \{-1\}$ and examine if it forms a group.

Associativity, existence of identity and existence of inverse are satisfied from the previous discussions but what about closure? We will fix $a \in \mathbb{Q} - \{-1\}$. Let us find the value of b for which we get $a \star b = -1$.

$$\begin{aligned} a \star b = -1 &\Rightarrow a + b + ab = -1 \\ &\Rightarrow a + b(1 + a) = -1 \\ &\Rightarrow b(1 + a) = -1 - a \\ &\Rightarrow b = \frac{-(1 + a)}{1 + a} = -1. \end{aligned}$$

Thus for any value of $a \in \mathbb{Q} - \{-1\}$, $b = -1$ will give us $a \star b = -1$ but this $b \notin \mathbb{Q} - \{-1\}$. Hence $(\mathbb{Q} - \{-1\}, \star)$ satisfies closure property as well. Therefore, $(\mathbb{Q} - \{-1\}, \star)$ forms a group.

***Example* 1.1.8:** Check $(\mathbb{Z}, \star)$ where $\star$ is defined as $a \star b = a + b + 2ab$ forms a group or not.

***Example* 1.1.9:** Let

$$A = \left\{ \frac{a}{b} \mid a \text{ and } b \text{ are co-prime and } 5 \nmid b \right\}.$$

Prove $(A, +)$ is a group.

Exercise

1. Let $A = \{\frac{a}{b} \mid a, b$ are co-prime and $5|b\}$. Then $(A, +)$ is

   (a) a group

   (b) a monoid

   (c) a magma

(d) none of the above

2. Let $\mathbb{Q}$ be the collection of all rational numbers. Then $(\mathbb{Q}, \times)$ is
   (a) a group
   (b) a monoid
   (c) only a magma
   (d) only a semi group

3. Let $\mathbb{Q}^*$ be the collection of all non-zero rational numbers. Then $(\mathbb{Q}^*, \times)$ is
   (a) a group
   (b) only a monoid
   (c) only a magma
   (d) none of the above

4. Let $\mathbb{Q}$ is the collection of all rationals, $\mathbb{Q}'$ is the collection of irrationals. Then $\mathbb{Q} + \mathbb{Q}'$ is
   (a) $\mathbb{Q}$
   (b) $\mathbb{Q}'$
   (c) $\mathbb{R}$
   (d) None of the above

5. Let $\mathbb{Q}'$ be the collection of irrational numbers. Then $(\mathbb{Q}', +)$ is
   (a) a group
   (b) a magma
   (c) a semi group
   (d) none of the above

6. Let E be the collection of all even integers. Then $(E, +)$ is

   (a) a group

   (b) a monoid

   (c) a magma

   (d) a semi group

7. Let $\mathbb{F}$ be the collection of all continuous functions on $[a, b]$ to $\mathbb{R}$. $+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ is defined by

$$(f + g)(x) = f(x) + g(x) \text{ for all } x \in [a, b].$$

   Then $(\mathbb{F}, +)$ is

   (a) a group

   (b) a monoid

   (c) a semigroup

   (d) an abelian group

8. Let $\mathbb{F}$ be the collection of all continuous functions on $[a, b]$ to $\mathbb{R}$ and $\circ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ be the composition of functions. Then the identity element in the group $(\mathbb{F}, \circ)$ is

   (a) $f(x) = 1$

   (b) $f(x) = 0$

   (c) $f(x) = x$

   (d) none of the above

9. Let $\mathbb{F}$ be the collection of all continuous functions on $[a, b]$ to $\mathbb{R}$ and $. : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ be defined by

$$(f.g)(x) = f(x)g(x) \quad \text{for all } x \in [a, b].$$

   Then $(\mathbb{F}, .)$ is

(a) a group

(b) a monoid

(c) a semigroup

(d) none of the above

Subgroups

Definition 1.1.3. Let $(G, \star)$ be a group and H is a subset of G. H is said to be a subgroup of G under $\star$ if $(H, \star)$ is itself a group.

***Note* 2:** For any group $(G, \star)$, there are two groups always associated with G. They are

1. $(G, \star)$, and
2. $(\{e\}, \star)$.

***Example* 1.1.10:** Let $(\mathbb{Z}, +)$ be a group. Consider the set of all even numbers as H. Clearly $H \subset \mathbb{Z}$, then $(H, +)$ is a subgroup of $(\mathbb{Z}, +)$.

For, let $a, b \in H$, that is, both a and b are even integers. Then clearly, $a + b$ is also an even integer, i.e., $a + b \in H$. Associativity holds trivially and 0 serves as an identity element.

For every $a \in H$, $-a$ is the inverse of a since

$$a = 2m \text{ for some } m \in \mathbb{Z} \Rightarrow -a = 2(-m).$$

Thus, the set of all even integers under addition forms a group and thus $(H, +)$ is a subgroup of $(\mathbb{Z}, +)$.

1.2 Cyclic groups

Observe the set of all even numbers

$$H = \{\cdots, -6, -4, -2, 0, 2, 4, 6, \cdots\}$$
$$H = \{\cdots, (2+2+2)^{-1}, (2+2)^{-1}, (2)^{-1}, 0, 1 \times 2, 2+2, 2+2+2, \cdots\},$$

since $4^{-1} = -4$ in $(\mathbb{Z}, +)$.

***Note* 3:** Generally, in every group $(G, \star)$,

$$a^2 = a \star a.$$

In particular, in $(\mathbb{Z}, +)$,

$$2^2 = 2 + 2, \quad 2^3 = 2 + 2 + 2.$$

Thus, 2^3 in $(\mathbb{Z}, +)$ does not mean $2 \times 2 \times 2$ but means $2 + 2 + 2$ and similarly $2^{-3} = -(2 + 2 + 2)$. Therefore,

$$H = \{\cdots, 2^{-3}, 2^{-2}, 2^{-1}, 0, 2^1, 2^2, 2^3, \cdots\}.$$

The above sort of representation is denoted by $\langle 2 \rangle$. That is,

$$H = \langle 2 \rangle, \text{ subgroup generated by } 2.$$

Definition 1.2.1 (Cyclic group)**.** A group $(G, \star)$ is said to be a cyclic group if there exists an element $a \in G$ such that $G = \langle a \rangle$, i.e., G is generated by a single element a.

***Note* 4:**

$$\langle a \rangle = \{\cdots, a^{-3}, a^{-2}, a^{-1}, e, a^1, a^2, a^3, \cdots\}$$
$$= \{\cdots, (a \star a \star a)^{-1}, (a \star a)^{-1}, (a)^{-1}, e, (a)^{-1}, (a \star a)^{-1}, (a \star a \star a)^{-1}, \cdots\}.$$

Definition 1.2.2 (Cyclic subgroup)**.** Let $(G, \star)$ be a group and let H be a subgroup of G. H is said to be a cyclic subgroup of G if there exists $a \in H$ such that $H = \langle a \rangle$.

***Note* 5:** For every group $(G, \star)$. Let $a \in G$. Then $\langle a \rangle$ is always a cyclic subgroup of $(G, \star)$.

Proof. Let $\langle a \rangle = \{\cdots, a^{-3}, a^{-2}, a^{-1}, e, a^1, a^2, a^3, \cdots\}$.

- Let $a^m, a^n \in \langle a \rangle$. Then

$$a^m \star a^n = \underbrace{a \star a \star \cdots \star a}_{\text{m times}} \star \underbrace{a \star a \star \cdots \star a}_{\text{n times}} = a^{m+n} \in \langle a \rangle.$$

  So closure property is satisfied.

-

$$\begin{aligned} (a^m \star a^n) \star a^s &= a^{m+n} \star a^s \\ &= a^{m+n+s} = a^{m+(n+s)} \\ &= a^m \star a^{n+s} \\ &= a^m \star (a^n \star a^s). \end{aligned}$$

  Hence associative axiom holds.

- Identity element e trivially exists.

- For every a^n, there exists $(a^n)^{-1} \in \langle a \rangle$ such that

$$a^n \star a^{-n} = a^{n+(-n)} = a^0 = e.$$

  Thus, inverse axiom is also satisfied.

- For any $a^n, a^m \in \langle a \rangle$,

$$a^n \star a^m = a^{n+m} = a^{m+n} = a^m \star a^n.$$

  Thus, commutativity axiom is satisfied.

Hence, $(\langle a \rangle, \star)$ is a group and in fact an abelian group. Therefore, $\langle a \rangle$ is a cyclic subgroup of $(G, \star)$. $\square$

Observation 1. From the above proof, we also observe that *every cyclic group is abelian* but the converse need not be true.

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MATHEMATICS

UNIT- II
Real Analysis

Your Success is Our Goal...

REAL ANALYSIS

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UNIT-2
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UNIT II: REAL ANALYSIS

SYLLABUS

Elementary set theory – Finite, countable and uncountable sets – Real number system as a complete ordered field – Archimedean Property – Supremum, infimum, Sequences and Series – Convergence – limit supremum – limit infimum – The Bolzano – Weierstrass theorem – The Heine – Borel Covering theorem – Continuity, Uniform Continuity, Differentiability – The Mean Value theorem for derivatives – Sequences and Series of functions – Uniform convergence. Riemann – Stieltjes integral: Definition and existence of the integral – properties of the integral – Integral and Differentaition – Integration of vector valued functions – Sequences and Series of functions: Uniform convergence – Continuity, Integration and Differentiation. Power series – Fourier series. Functions of several variables – Directional derivative – Partial derivative – derivative as a linear transformation – The Inverse function theorem and The Implicit function theorem

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Chapter 1

Basics and Metric Spaces

1.1 Elementary set theory

Definition 1.1.1 (Relations)**.** Let A and B be two non-empty sets, a relation from A to B is a subset of the Cartesian product $A \times B$. That is, any subset of $A \times B$ is said to be a relation from A to B, and we call any subset of $A \times A$ as a relation on A.

Let R be a relation $A \to B$.

1. If $R = \phi$, then it is called the empty relation.
2. If $R = A \times B$, it is the universal relation.
3. If $A = B$, and if $R = \{(a, a);\ \forall a \in A\}$, is called the identity relation.

Classification of Relations: Let A be a non-empty set, and R be a relation on A, then

1. R is said to be reflexive if $(a, a) \in R,\ \forall a \in A$
2. R is said to be symmetric if $(a, b) \in R \Rightarrow (b, a) \in R, \forall a, b \in A$
3. R said to be transitive if (a, b) and $(b, c) \in R \Rightarrow (a, c) \in R$
4. R is said to be an equivalence relation on A, if R is said to be reflexive, symmetric & transitive

5. R is said to be antisymmetric, if both (a, b) and $(b, a) \in R$, iff $a = b$

6. R is said to be a partially ordered relation, if R is reflexive, antisymmetric & transitive

Definition 1.1.2 (Equivalence Class)**.** Let R be an equivalence relation on a non-empty set A, then there exist a partition of A into disjoint classes, called equivalence classes.

Definition 1.1.3. For $a \in A$, the equivalence class of a is given by $[a] = \{b \in A;\ (a, b) \in R$ or $aRb\}$.

Properties:

1. $a \in [a]$

2. If $b \in [a]$, then $[a] = [b]$

3. If $b \notin [a]$, then $[a] \cap [b] = \phi$

Result: Let A & B be two non-empty finite sets, with $|A| = n,\ |B| = m$, then

1. No.of relations from A to B is 2^{mn}

2. No.of relations on A is 2^{n^2}

3. No.of reflexive relations on A is 2^{n^2-n}

4. No.of symmetric relations on A is $2^{n(n+1)/2}$

5. No.of relations on A which are both reflexive & symmetric $2^{(n^2-n)/2}$

6. Number of equivalence relation on A (= Number of partition on A) is given by $B_n = \sum\limits_{k=0}^{n-1} {}^{n-1}C_k B_k$, where $B_0 = 1$.

Definition 1.1.4 (Mapping)**.** Consider two sets A and B whose elements may be any objects and suppose that with each element x of A, there is associated an element of B which we denote by $f(x)$. Then f is said to be a *function* from A to B (or a *mapping* of A into B). The set A is called the *domain* of f (or we say f is defined on A), and the elements $f(x)$ are called the *values* of f. The set of all values of f is called *range* of f.

Definition 1.1.5 (Image)**.** Let A and B be two sets and let f be a mapping of A into B. If $E \subset A$, $f(E)$ is defined to the set of all elements $f(x)$, for $x \in E$. The set $f(E)$ is called the *image* of E under f.

Remark 1.1.1. It is clear that $f(A) \subset B$; if $f(A) = B$, then the map f is said to be an onto function.

Definition 1.1.6 (Inverse image)**.** If $E \subset B$, $f^{-1}(E)$ denotes the set of all $x \in A$ such that $f(x) \in E$. The set $f^{-1}(E)$ is the inverse image of E under f.

Definition 1.1.7 (One-one mapping)**.** If for each $y \in B$, $f^{-1}(y)$ consist of atmost one element of A, then f is said to be one-one mapping of A into B. (or) If $x_1, x_2 \in A$, then the function is said to be one-one mapping if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. (or) $f(x_1) = f(x_2)$ implies $x_1 \neq x_2$.

Definition 1.1.8 (One-one correspondence)**.** If there exists a one-one mapping of A onto B, we say that A and B are in one-one correspondence (or) that A and B have the same cardinal number (or) that A and B are equivalent, and we write $A \sim B$.
This relation has the following properties:

1. $A \sim B$ (reflexive)

2. If $A \sim B$, then $B \sim A$ (symmetric)

3. If $A \sim B$ and $B \sim C$, then $A \sim C$ (transitive)

Any relation with these properties is called an equivalence relation.

Definition 1.1.9. Let A and Ω be the sets and suppose that E_α is the subset of Ω associated with A, that is, $\{E_\alpha\}$ where $\alpha \in A$.

The union of sets E_α is defined to be the set S such that $x \in S$ if and only if $x \in E_\alpha$ for atleast one $\alpha \in A$.

$$S = \cup_{\alpha \in A} E_\alpha$$

If A consists of integers $1, 2, \cdots, n$, then the set

$$S = \cup_{m=1}^{n} E_m.$$

If A is the set of postive integers, then

$$S = \cup_{m=1}^{\infty} E_m.$$

Similarly, the intersection of sets E_α is defined to be the set T such that $x \in T$ if and only if $x \in E_\alpha$ for all $\alpha \in A$.

$$T = \cap_{\alpha \in A} E_\alpha$$

If A consists of integers $1, 2, \cdots, n$, then the set

$$T = \cap_{m=1}^{n} E_m.$$

If A is the set of positive integers, then

$$T = \cap_{m=1}^{\infty} E_m.$$

Remark 1.1.2. If $A \cap B$ is not empty, we say that A and B intersect; otherwise they are disjoint.

***Problem* 1.1.1:** Let $E_1 = \{1, 2, 3\}$, $E_2 = \{2, 3, 4\}$. Then $E_1 \cup E_2 = \{1, 2, 3, 4\}$, $E_1 \cap E_2 = \{2, 3\}$.

***Problem* 1.1.2:** Let $A = \{x | x \in \mathbb{R} \quad \& \quad 0 < x \leq 1\}$ and $E_x = \{y | y \in \mathbb{R} \quad \& \quad 0 < y < x\}$. Then $\cup_{x \in A} E_x = E_1$ and $\cap_{x \in A} E_x = \phi$.

Result 1.1.1.

1. $A \cup B = B \cup A$
2. $A \cap B = B \cap A$
3. $(A \cup B) \cup C = A \cup (B \cup C)$
4. $(A \cap B) \cap C = A \cap (B \cap C)$
5. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
6. $A \subset A \cup B$
7. $A \cap B \subset A$
8. If $A \subset B$, then $A \cup B = B, A \cap B = A$

9. $A \cup \phi = A$

10. $A \cap \phi = \phi$

Result 1.1.2. Let $f : A \to B$ be a function where A and B be two non-empty finite sets with $n(A)$=n and $n(B)$=m then

1. Number of functions from A to $B = m^n$

2. Number of one-one function from A to B
$$= \begin{cases} {}^mP_n, & m \geqslant n \\ 0, & m < n \end{cases}$$

3. Number of onto function from A to B
$$= \begin{cases} \sum\limits_{i=1}^{m}(-1)^{m-i}({}^mC_i i^n), & m \leq n \\ \quad 0, & m > n \end{cases} = \begin{cases} \sum\limits_{r=0}^{m-1}(-1)^r({}^mC_r)(m-r)^n, & m \leq n \\ \quad 0, & m > n \end{cases}$$
If $m = 2$, then Number of onto function $f : A \to B$ is $2^n - 2$.

4. Number of bijective function from A to B is $\begin{cases} n!, & m = n \\ 0, & m \neq n \end{cases}$

5. If $m = n$, then $f : A \to B$ is one-one $\implies$ f is onto.

Definition 1.1.10 (Compositions of functions)**.** Let A, B, C be three non-empty sets, $f : A \to B$ and $g : B \to C$ be two functions, then we can define a function A to C, $gof = A \to C$ by $(gof)(x) = g(f(x)),\ \forall x \in A$.

If $A = C$, then both the compositions gof & fog are defined.

If $A = B = C$, then the compositions gof, fog, fof, gog etc are also defined.

Result 1.1.3. 1. $fog \neq gof$, in general.

2. If f & g are one-one, then gof and fog are also one-one, provided they exists.

3. If f & g are onto, then gof and fog are also onto, provided they exists.

4. If gof is one-one, then f is one-one.

5. If gof is onto, then g is onto.

Definition 1.1.11 (Inverse of Functions)**.** Let A & B be two non-empty sets, and let $f : A \to B$ be a function, f is said to have a left inverse if there exist a function $g : B \to A$ such that $gof = I_A$. f is said to have a right inverse if there exist a function $g : B \to A$ such that $fog = I_B$. f is said to be invertible if there exist a function $g : B \to A$ such that $gof : I_A$ and $fog = I_B$. If f is invertible, then we write $g = f^{-1}$.

Result 1.1.4. 1. A function $f : A \to B$ is invertible, if f is both one-one and onto.

2. If $f : A \to B$ is one-one then it has a left inverse.

3. If $f : A \to B$ is onto then it has a right inverse.

4. Let A and B be two non-empty sets, if there exist a bijection between A and B then A and B are said to be numerically equivalent.

5. Two finite sets A and B are numerically equivalent iff $n(A) = n(B)$.

Definition 1.1.12 (Even and Odd Functions)**.** Let f be a real function, f is said to be an even function if $f(-x) = f(x),\ \forall x \in D(f)$ and f is said to be an odd function if $f(-x) = -f(x),\ \forall x \in D(f)$.

Result 1.1.5. 1. $f(x) = 0$ is the only function which is both even and odd.

2. Non-zero constant functions are even functions.

3. Sum of two even functions is even.

4. Difference of two even functions is even.

5. Sum or difference of two odd functions is odd.

6. Sum of an even function with an odd function need not be even or odd.

7. Difference of an even function with an odd function need not be even or odd.
8. Product of two even functions is even.
9. Product of two odd functions is even.
10. Product of an even function with an odd function is odd.
11. Composition of two even functions is even.
12. Composition of two odd functions is odd.
13. gof is even if f is even.
14. Composition of two functions is even if atleast one of them is even.
15. Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary function,then, $f(x) + f(-x)$ is an even function and $f(x) - f(-x)$ is an odd function
    Also, $f(x) = \left(\frac{f(x)+f(-x)}{2}\right) + \left(\frac{f(x)-f(-x)}{2}\right)$,
    i.e., any function can be expressed as the sum of an even and an odd function.

Definition 1.1.13 (Periodic Functions)**.** Let f be a real function, f is said to be periodic, if $f(x + T) = f(x)$, for some $T \in \mathbb{R}$, then, T is called the period of f.

***Example* 1.1.1:**

1. The trigonometric function $\sin x, \cos x, \operatorname{cosec} x, \sec x$ are periodic with period $2k\pi, k \in \mathbb{Z}$ and fundamental period is 2π.
2. The trigonometric function $\tan x$ and $\cot x$ are periodic with period $k\pi, k \in \mathbb{Z}$, their fundamental period is π.
3. Fundamental period of the function $\sin^n x, \cos^n x, \sec^n x, \operatorname{cosec}^n x$ is π if n is even and 2π if n is odd.
4. Fundamental period of $\tan^n x,\ \cot^n x$ is $\pi,\ \forall n$.

   Fundamental period of $\sin(ax),\ a \neq 0$ is $\left(\frac{2\pi}{a}\right)$. In general let f be a periodic function with fundamental period T, then the fundamental period of the function $f(ax)$ is $\frac{T}{a}$.

5. The function $\{x\} = x - [x]$ is periodic with period $k,\ k \in \mathbb{Z}$ and fundamental period is 1.

6. Constant functions are periodic with period $a,\ a \in \mathbb{R}$, and does not have fundamental periods.

Result 1.1.6. Let f & g be two periodic functions with periods T_1 & T_2 respectively, then $f+g, f-g, fg$ are also periodic if there exist positive integers a & b such that $aT_1 = bT_2$. If $f+g, f-g$ and fg are periodic, then period of these function $\leqslant LCM(T_1, T_2)$.

***Note* 1:** Sum of two periodic functions need not be periodic.

***Example* 1.1.2:** $\sin x, \sin 2x$ are periodic with fundamental periods 2π and 1 respectively. But their sum is not period, as there does not exist positive integers such that $a \cdot 2\pi = b \cdot 1$.

Definition 1.1.14 (Monotone Functions)**.** f is said to be monotone if it is either monotonically increasing or monotonically decreasing.

1. If $x < y \Rightarrow f(x) \leq f(y),\ \forall\, x, y \in D(f)$, then f is said to be monotonically increasing.

2. If $x < y \Rightarrow f(x) \geq f(y),\ \forall\, x, y \in D(f)$, then f is said to be monotonically decreasing.

3. If $x < y \Rightarrow f(x) < f(y),\ \forall\ x, y \in D(f)$, then f is said to be strictly monotonically increasing.

4. If $x < y \Rightarrow f(x) > f(y),\ \forall\ x, y \in D(f)$, then f is said to be strictly monotonically decreasing.

Result 1.1.7. 1. Constant functions are both monotonically increasing and monotonically decreasing. They are usually considered as non-decreasing functions.

2. $f(x) = x, e^x, \log_a x,\ x \in (0, \infty)$ are monotonically increasing functions.

3. $f(x) = \frac{1}{x}$ is monotonically decreasing in the intervals $(-\infty, 0), (0, \infty)$.

4. Lines with positive slope are monotonically increasing and the lines with negative slope are monotonically decreasing.

5. Let $f : A \to B,\ g : C \to B$ be two real functions, then the function $f+g, f-g, fg$ are defined only if $A \cap C \neq \phi$. If $A \cap C \neq \phi$, then the domains of $f+g, f-g, fg$ are $A \cap C$ itself.

   The $D\left(\frac{f}{g}\right)$ is given by $A \cap C \setminus \{x \in C \mid g(x) = 0\}$

Definition 1.1.15 (Bounded Functions)**.** Let f be a real function, f is said to be bounded, if there exist a positive number, such that $|f(x)| \leq M,\ \forall x \in D(f)$.

***Note* 2** (Partial fractions)**:** If $a < b,\ \frac{1}{(x-a)(x-b)} = \frac{1}{(b-a)}\left[\frac{1}{(x-a)} - \frac{1}{(x-b)}\right]$

1.2 Countable sets

Definition 1.2.1. For any positive integer n, let J_n be the set whose elements are the integers $1, 2, \cdots, n$. Let J be the set consisting of all positive integers. For any set A, we define

1. A is finite, if $A \sim J_n$ for some n.
2. A is infinite, if A is not finite.
3. A is countable, if $A \sim J$.
4. A is uncountable, if A is neither finite nor countable.
5. A is atmost countable, if A is finite or countable.

Note: Countable sets are also called enumerable or denumerable.

***Problem* 1.2.1** (Integers are countable)**:** Let A be the set of all integers. Consider the following arrangement of the sets A and J.

$A : 0, 1, -1, 2, -2, \cdots$

$J : 1, 2, 3, \cdots.$

Define the function $f : J \to A$ by

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ -\frac{(n-1)}{2}, & \text{if } n \text{ is odd} \end{cases}.$$

This function f is one-one and onto. Therefore, the set A is countable.

Remark 1.2.1. A finite set cannot be equivalent to one of its proper subsets; A is infinite if A is equivalent to one of its proper subsets.

***Example* 1.2.1:** By previous remark, we have proved that $J \sim \mathbb{Z}$ but we know that J is a proper subset of $\mathbb{Z}$.

Remark 1.2.2. Let X be a countable infinite set. Then we write $|X| = \aleph_0$. (Read as "Aleph not"). If X is finite, say n number of distinct elements, then $|X| = n$.

Result 1.2.1. Every countable infinite set is equivalent to a proper subset of itself.

Proof. The proof is by axiom of choice. Let $X = \{x_1, x_2, \cdots\}$ and $Y = \{x_2, x_3, \cdots\} \subset X$. Define $f : X \to Y$ by $f(x_n) = x_{n+1}$.

Clearly, f is one-one and onto. Thus, we proved $X \sim Y$, even though Y is a proper subset of X. □

Result 1.2.2. Every infinite set has a countable infinite subset. Consequently,

$$A \text{ is infinite set} \Rightarrow |A| \geq \aleph_0.$$

Proof. Let X be an infinite set. Then either

1. X is countable infinite (or)

   By previous result, it has a countable infinite subset.

2. X is uncountable infinite

   By axoim of choice, we are able to choose $Y = \{x_1, x_2, \cdots\} \subset X$, which is a countable infinite subset.

□

Result 1.2.3. Every infinite set is equivalent to one of its proper subsets.

Proof. Now we are in a situation to prove every infinite set is equivalent to one of its proper subset.

Case (i) If X is countable, then by result 1.2.1, there exists $Y \subset X$ such that $Y \sim X$.

Case (ii) Let X be an uncountable infinite set. Then there exists a countable infinite subset X_1 of X. Put $X_2 = X - X_1$. Then

$$X_1 \cap X_2 = \emptyset \quad \text{and} \quad x_1 \cup X_2 = X.$$

Here X_1 is a countable infinite set, and so $X_1 \sim X_3$ when $X_3 \subset X_1$. So we have mapped $X_1 \cap X_3$ and $X_2 \cap X_2$. Then, $X_3 \cup X_2 = Y$. We have

$$X \sim Y \quad \text{and} \quad Y \subset X.$$

Hence proved.

□

Remark 1.2.3. This is the characterization of an infinite set.

***Example* 1.2.2:** Are there infinite set whose cardinality is greater than $\aleph_0$?

Let $|X| = 3$. Then $\mathscr{P}(X)$ is the collection of all subsets of X.

$$|\mathscr{P}(X)| = 2^3 = 8.$$

So clearly, $|\mathscr{P}(X)| > |X|$.

Remark 1.2.4. If X is infinite, then $|\mathscr{P}(X)| > |X|$.

Result 1.2.4. There is no onto map from X to $\mathscr{P}(X)$.

By using this result, we can say $|\mathscr{P}(\mathbb{N})| > |\mathbb{N}|$.

$$\implies |\mathscr{P}(\mathbb{N})| = 2^{|\mathbb{N}|} = 2^{\aleph_0}.$$
$$\implies 2^{\aleph_0} > \aleph_0.$$

Put $\mathfrak{c} = 2^{\aleph_0}$. (Continum)

Different characterization of $2^{\aleph_0}$: Let X and Y be two finite sets such that $|X| = m$ and $|Y| = n$. Put

$$\mathcal{F} = \{f : X \to Y\} = \text{ Collection of all functions from } X \text{ to } Y.$$

What is the cardinality of $\mathcal{F}$, i.e., $|\mathcal{F}| =$?

For each $x \in X$, there are n choices. Thus, totally we have n^m choices. That is,

$$|\mathcal{F}| = n^m = |Y|^{|X|}.$$

Now, what is the cardinality of $\mathcal{F}_{\{0,1\}}$?

- $\mathcal{F}_{\{0,1\}}$ is the collection of all functions from $\mathbb{N}$ to $\{0, 1\}$. Then,

$$|\mathcal{F}_{\{0,1\}}| = 2^{|\mathbb{N}|} = 2^{\aleph_0} = \mathfrak{c}.$$

Result 1.2.5. Clearly, $\mathscr{P}(\mathbb{N}) \sim \mathcal{F}_{\{0,1\}}$.

- We know that every real number in $[0, 1]$ have a binary representation. Let $f \in \mathcal{F}_{\{0,1\}}$. Then f can be viewed as a sequence. Therefore, every $x \in [0, 1]$ can be associated with some $f \in \mathcal{F}_{\{0,1\}}$ and hence

$$\mathcal{F}_{\{0,1\}} \sim [0, 1].$$

So, $|[0, 1]| = 2^{\aleph_0}$.

- We know that $[0, 1] \sim \mathbb{R}$.

$$[0,1] \underset{f_1}{\overset{\frown}{\to}} \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \underset{f_2=\tan^{-1} x}{\overset{\frown}{\to}} \mathbb{R}.$$

Here f_1 is a linear map, both f_1 and f_2 are one-one and onto and thus their composition is also a one-one onto map. So, $[0, 1] \sim \mathbb{R}$. Thus,

$$|\mathbb{R}| = 2^{\aleph_0} = \mathfrak{c}.$$

Thus, we have obtained 3 different characterizations of $2^{\aleph_0}$.

Result 1.2.6 (Hilbert Unproven Conjecture)**.** There is no other infinities between $\aleph_0$ and $2^{\aleph_0} = \mathfrak{c}$. That is, if there is an infinity between $\aleph_0$ and $2^{\aleph_0}$, it must be equal to either one of them.

$$\aleph_0 \leq z \leq 2^{\aleph_0} \implies z = \aleph_0 \text{ or } z = 2^{\aleph_0}.$$

Note: Even though it is unproven, we use this result and there are questions from this result.

We now pose a **question**. *Are there infinities greater than* $2^{\aleph_0}$*?* The answer is YES. Look at $\mathscr{P}(\mathbb{R})$ and use Result 1.2.4. That is,

we have no ONTO function from $\mathbb{R}$ to $\mathscr{P}(\mathbb{R})$.

So $2^{\aleph_0} < 2^{2^{\aleph_0}}$, i.e., $\mathfrak{c} < 2^{\mathfrak{c}}$. Likewise

$$\mathfrak{c} < 2^{\mathfrak{c}} < 2^{2^{\mathfrak{c}}} < 2^{2^{2^{\mathfrak{c}}}} < \cdots .$$

Let $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$, $\cdots$. Then we have

$$\aleph_0 < \aleph_1 < \aleph_2 < \cdots .$$

1.3 Arithmetic of cardinalities

Let $X = \{a, 2, 3\}$ and $Y = \{1, b, c\}$. Then

$$X \cap Y = \emptyset, \quad |X| = 3, \quad \text{and} \quad |Y| = 3.$$

Here $|X \cup Y| = |\{a, b, c, 1, 2, 3\}| = 6$ and $|X \cap Y| = |\{\}| = 0$. Thus,

$$|X \cup Y| = |X| + |Y| \quad \text{and} \quad |X \cap Y| = |X| - |Y|.$$

We are able to see that there are some arithmetic in cardinalities. We will improve this to infinite sets.

Definition 1.3.1. If X and Y are disjoint, then $|X| + |Y| = |X \cup Y|$.

The next natural question is to associate $|X| \times |Y|$. Let us consider the next definition which will answer this question.

Definition 1.3.2. $|X| \times |Y| = |X \times Y|$.

Definition 1.3.3. $|X|^{|Y|} = |\mathcal{F}_{\{0,1\}}|$, where $|\mathcal{F}_{\{0,1\}}$ is the collection of all functions from Y to X.

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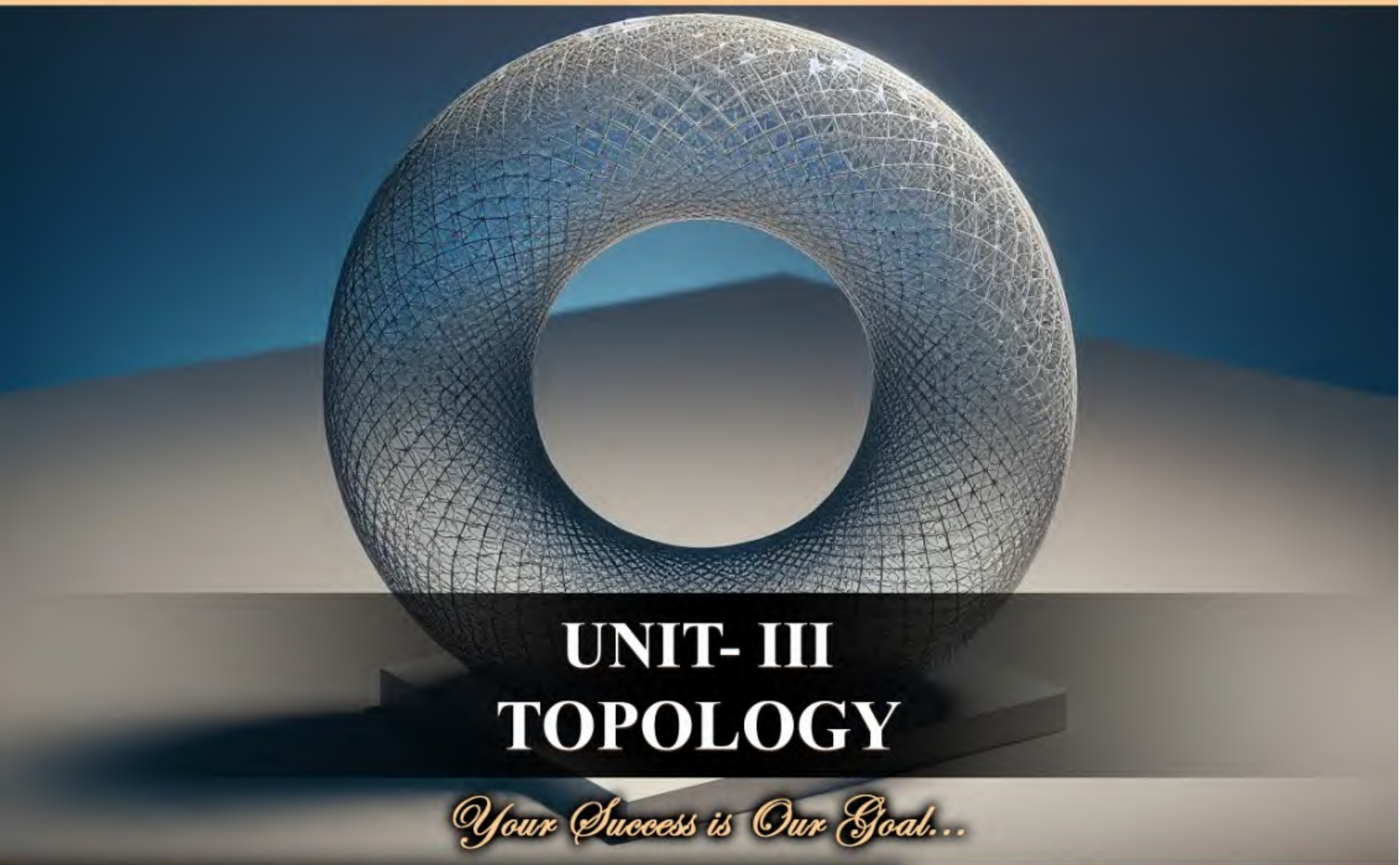

TOPOLOGY

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UNIT-III

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UNIT – III

TOPOLOGY

UNIT III - TOPOLOGY

SYLLABUS:

Topological spaces – Basis – The order Topology – The product Topology – The subspace Topology – Closed sets and limit points. Continuous functions – The box and product Topologies – The matrix Topology. Connected spaces – Connected subspaces of the real line – Components and local connectedness – compact spaces – Compact subspaces of the real line – Limit point compactness – Local compactness. Countability and separation Axioms – Normal spaces – The Urysohn Lemma – The Urysohn metrization theorem – The Tietze extension theorem.

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3.1. TOPOLOGICAL SPACES

3.1.1 TOPOLOGY DEFINITION

- A topology on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:
  1) ø and X are in $\mathcal{T}$.
  2) The union of the elements of any sub collection of $\mathcal{T}$ is in $\mathcal{T}$.
  3) The intersection of the elements of any finite sub collection of $\mathcal{T}$ is in $\mathcal{T}$.

- A set X for which a topology $\mathcal{T}$ has been specified is called a topological space.

Note:

1) A topological space is an ordered pair (X, $\mathcal{T}$) consisting of a set X and a topology $\mathcal{T}$ on X.
2) If X is a topological space with topology $\mathcal{T}$, we say that a subset U of X is an open set of X if U belongs to the collection $\mathcal{T}$.
3) Using this terminology, we can say that a topological space is a set X together with a collection of subsets of X, called open sets, such that $\emptyset$ and X are both open, and such that arbitrary unions and finite intersections of open sets are open.

Example 1.

- Let X be a three-element set, X = {a, b, c}. There are many possible topologies on X.
- From this example, you can see that even a three-element set has many different topologies. But not every collection of subsets of X is a topology on X.

Example 2.

- If X is any set, the collection of all subsets of X is a topology on X; it is called the discrete topology. The collection consisting of X and ø only is also a topology on X; we shall call it the indiscrete topology, or the trivial topology.

Example 3.

- Let X be a set; let $\mathcal{T}_f$ be the collection of all subsets U of X such that X –U either is finite or is all of X. Then $\mathcal{T}_f$ is a topology on X, called the infinite complement topology. Both X and ø are in $\mathcal{T}_f$, since X – X is finite and X – ø is all of X.
- If $\{U_\alpha\}$ is an indexed family of nonempty elements of $\mathcal{T}_f$, to show that $\cup U_\alpha$ is in $\mathcal{T}_f$, we compute

$$X - \bigcup U_\alpha = \bigcap (X - U_\alpha)$$

- The latter set is finite because each set X - U_α is finite. If $U_l,\dots,U_n$ are nonempty elements of $\mathcal{T}_f$, to show that $\cap U_i$ is in $\mathcal{T}_f$, we compute.
- The latter set is a finite union of finite sets and, therefore, finite.

Example. 4

- Let X be a set; let $\mathcal{T}_c$ be the collection of all subsets U of X such that X - U either is countable or is all of X. Then $\mathcal{T}_c$ is a topology on X.

3.1.2. DEFINITION

- Suppose that $\mathcal{T}$ and $\mathcal{T}'$ a are two topologies on a given set X. If $\mathcal{T} \supset \mathcal{T}'$, we say that $\mathcal{T}'$ is finer than $\mathcal{T}$; if $\mathcal{T}'$ properly contains $\mathcal{T}$, we say that $\mathcal{T}'$ is strictly finer than $\mathcal{T}$. We also say that $\mathcal{T}$ is coarser than $\mathcal{T}'$, or strictly coarser, in these two respective situations. We say $\mathcal{T}$ is comparable with $\mathcal{T}'$ if either $\mathcal{T}' \supset \mathcal{T}$ or $\mathcal{T} \supset \mathcal{T}'$.

Note:

- If $\mathcal{T} \supset \mathcal{T}'$, then we say that $\mathcal{T}'$ is larger than $\mathcal{T}$, and $\mathcal{T}$ is smaller than $\mathcal{T}'$. This is certainly acceptable terminology, the words "finer" and "coarser."

3.2. BASIS FOR A TOPOLOGY

3.2.1. BASIS DEFINITION:

- If X is a set, a basis for a topology on X is a collection $\mathfrak{B}$ of subsets of X (called basis elements) such that
  1) For each x $\in$ X, there is at least one basis element $\mathfrak{B}$ containing x.
  2) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.
- If $\mathfrak{B}$ satisfies these two conditions, then we define the topology $\mathcal{T}$ generated by $\mathfrak{B}$ as follows: A subset U of X is said to be open in X (that is, to be an element of $\mathcal{T}$) if for each x $\in$ U, there is a basis element B$\in \mathfrak{B}$ such that x $\in$ B and B $\subset$ U. Note that each basis element is itself an element of $\mathcal{T}$.

Example 1.

- Let $\mathfrak{B}$ be the collection of all circular regions (interiors of circles) in the plane. Then $\mathfrak{B}$ satisfies both the conditions for a basis. In the topology generated by $\mathfrak{B}$, a subset U of the plane is open if every x in U lies in some circular region contained in U.

Example 2.

- Let $\mathfrak{B}'$ be the collection of all rectangular regions (interiors of rectangles) in the plane, where the rectangles have sides parallel to the coordinate axes. Then $\mathfrak{B}'$ satisfies both conditions for a basis. The second condition is trivial, because the intersection of any two basis elements is itself a basis element (or empty). As we shall see later, the basis $\mathfrak{B}'$ generates the same topology on the plane as the basis $\mathfrak{B}$ given in the preceding example.

Example 3.

- If X is any set, the collection of all one-point subsets of X is a basis for the discrete topology on X.

Result:

1) Let us check now that the collection $\mathcal{T}$ generated by the basis $\mathfrak{B}$ is, in fact, a topology on X. If U is the empty set, it satisfies the defining condition of openness vacuously. Likewise, X is in $\mathcal{T}$, since for each $x \in$ X there is some basis element B containing x and contained in X. Now let us take an indexed family $\{U_\alpha\}_{\alpha \in J}$, of elements of $\mathcal{T}$ and show that

$$U = \bigcup_{\alpha \in J} U_\alpha$$

belongs to $\mathcal{T}$. Given $x \in$ U, there is an index α such that $x \in U_\alpha$. Since U_α is open, there is a basis element B such that $x \in B \subset U_\alpha$. Then $x \in B$ and $B \subset U$, so that U is open, by definition.

2) Now let us take two elements U_1 and U_2 of $\mathcal{T}$ and show that $U_1 \cap U_2$ belongs to $\mathcal{T}$. Given $x \in U_1 \cap U_2$, choose a basis element B_1 containing x such that $B_1 \subset U_1$; choose also a basis element B_2 containing x such that $B_2 \subset U_2$.

3) The second condition for a basis enables us to choose a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$. Then $x \in B_3$ and $B_3 \subset U_1 \cap U_2$, so $U_1 \cap U_2$belongs to $\mathcal{T}$, by definition.

4) Finally, we show by induction that any finite intersection $U_1 \cap \ldots\ldots \cap U_n$ of elements of $\mathcal{T}$ is in $\mathcal{T}$. This fact is trivial for n = 1; we suppose it true for n - 1 and prove it for n. Now

$$(U_1 \cap ... \cap U_n) = (U_1 \cap ... \cap U_{n-1}) \cap U_n$$

- By hypothesis, $U_1 \cap \ldots\ldots \cap U_{n-1}$ belongs to $\mathcal{T}$; by the result just proved, the inter- section of $U_1 \cap \ldots\ldots \cap U_{n-1}$ and U_n also belongs to $\mathcal{T}$.

Thus we have checked that collection of open sets generated by a basis $\mathfrak{B}$ is a topology.

3.2.2. LEMMA

- Let X be a set; Jet $\mathfrak{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathfrak{B}$.

Proof:

- Given a collection of elements of $\mathfrak{B}$, they are also elements of $\mathcal{T}$. Because $\mathcal{T}$ is a topology, their union is in $\mathcal{T}$. Conversely, given $U \in \mathcal{T}$, choose for each $x \in U$ an element B_x of $\mathfrak{B}$ such that $x \in B_x \subset U$. Then $U = \bigcup_{x \in U} B_x$, so U equals a union of elements of $\mathfrak{B}$.
- This lemma states that every open set V in X can be expressed as a union of basic elements.

3.2.3. LEMMA

- Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X.

Proof:

- We must show that $\mathcal{C}$ is a basis. The first condition for a basis is easy: Given $x \in X$, since X is itself an open set, there is by hypothesis an element C of $\mathcal{C}$ such that $x \in C \subset X$. To check the second condition, let x belong to $C_1 \cap C_2$, where C_1 and C_2 are elements of $\mathcal{C}$. Since C_1 and C_2 are open, so is $C_1 \cap C_2$. Therefore, there exists by hypothesis an element C_3 in $\mathcal{C}$ such that $x \in C_3 \subset C_1 \cap C_2$.
- Let $\mathcal{T}$ be the collection of open sets of X; we must show that the topology $\mathcal{T}'$ generated by $\mathcal{C}$ equals the topology $\mathcal{T}$. First, note that if U belongs to $\mathcal{T}$ and if $x \in U$, then there is by hypothesis an element C of $\mathcal{C}$ such that $x \in C \subset U$. It follows that U belongs to the topology $\mathcal{T}'$, by definition. Conversely, if W belongs to the topology $\mathcal{T}'$, then W equals a union of elements of $\mathcal{C}$, by the preceding lemma. Since each element of C belongs to $\mathcal{T}$ and $\mathcal{T}$ is a topology, W also belongs to $\mathcal{T}$.

3.2.4. LEMMA

- Let $\mathfrak{B}$ and $\mathfrak{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively, on X. Then the following are equivalent:
    1) $\mathcal{T}'$ is finer than $\mathcal{T}$.
    2) For each $x \in X$ and each basis element $B \in \mathfrak{B}$ containing x, there is a basis element $B' \in \mathfrak{B}'$ such that $x \in B' \subset B$.

Proof:

- (2)⟹(1). Given an element U of $\mathcal{T}$, we wish to show that $U \in \mathcal{T}'$. Let $x \in U$. Since $\mathfrak{B}$ generates $\mathcal{T}$, there is an element $B \in \mathfrak{B}'$ such that $x \in B \subset U$. Condition (2) tells us there exists an element $B' \in \mathfrak{B}'$ such that $x \in B' \subset B$. Then $x \in B' \subset U$, so $U \in \mathcal{T}$, by definition.
- (1)⟹(2). We are given $x \in X$ and $B \in \mathfrak{B}$, with $x \in B$. Now B belongs to $\mathcal{T}$ by definition and $\mathcal{T} \subset \mathcal{T}'$ by condition (l); therefore, $B \in \mathcal{T}'$. Since $\mathcal{T}'$ is generated by $\mathfrak{B}'$, there is an element $B' \in \mathfrak{B}'$ such that $x \in B' \subset B$.

Example 4.

- The collection $\mathfrak{B}$ of all circular regions in the plane generates the same topology as the collection $\mathfrak{B}'$ of all rectangular regions.

3.2.5. STANDARD TOPOLOGY DEFINITION:

- If $\mathfrak{B}$ is the collection of all open intervals in the real line, $(a, b) = \{x / a < x < b\}$, the topology generated by $\mathfrak{B}$ is called the **standard topology** on the real line. Whenever we consider $\mathbb{R}$, we shall suppose it is given this topology unless we specifically state otherwise.

3.2.6. LOWER LIMIT TOPOLOGY DEFINITION:

- If $\mathfrak{B}'$ is the collection of all half-open intervals of the form $[a, b) = \{x / a \leq x < b\}$, where $a < b$, the topology generated by $\mathfrak{B}'$ is called the **lower limit topology** on $\mathbb{R}$ When $\mathbb{R}$ is given the lower limit topology, we denote it by $\mathbb{R}_l$.
- Finally let K denote the set of all numbers of the form $1/n$, for $n \in Z_+$, and

3.2.7. K-TOPOLOGY DEFINITION:

- Let $\mathfrak{B}''$ be the collection of all open intervals (a, b), along with all sets of the form (a, b) - K. The topology generated by $\mathfrak{B}''$ will be called the **K-topology** on $\mathbb{R}$. When $\mathbb{R}$ is given this topology, we denote it by $\mathbb{R}_k$.
- It is easy to see that all three of these collections are bases; in each case, the intersection of two basis elements is either another basis element or is empty. The relation between these topologies is the following:

3.2.8. LEMMA

- The topologies $\mathbb{R}_l$ and $\mathbb{R}_k$ are strictly finer than the standard topology on $\mathbb{R}$, but are not comparable with one another.

Proof:

- Let $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$ be the topologies of $\mathbb{R}$, $\mathbb{R}_l$ and $\mathbb{R}_k$, respectively. Given a basis element (a, b) for $\mathcal{T}$ and a point x of (a ,b), the basis element [x, b) for $\mathcal{T}'$ contains x and lies in (a, b). On the other hand, given the basis element [x, d) for $\mathcal{T}'$, there is no open interval (a, b) that contains x and lies in [x, d). Thus $\mathcal{T}'$ is strictly finer than $\mathcal{T}$.
- A similar argument applies to $\mathbb{R}_k$. Given a basis element (a, b) for $\mathcal{T}$ and a point x of (a, b), this same interval is a basis element for $\mathcal{T}''$ that contains x. On the other hand, given the basis element B = (-1, 1) - K for $\mathcal{T}''$ and the point 0 of B, there is no open interval that contains 0 and lies in B.
- We leave it to you to show that the topologies of $\mathbb{R}_l$ and $\mathbb{R}_k$ are not comparable.

3.2.9. SUBBASIS DEFINITION

- A subbasis S for a topology on X is a collection of subsets of X whose union equals X. The **topology generated by the subbasis** S is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of S.

Result:

1) We must of course check that $\mathcal{T}$ is a topology. For this purpose it will suffice to show that the collection $\mathfrak{B}$ of all finite intersections of elements of S is a basis, for then the collection $\mathcal{T}$ of all unions of elements of $\mathfrak{B}$ is a topology, by Lemma 3.2.2. Given x $\in$ X, it belongs to an element of S and hence to an element of $\mathfrak{B}$; this is the first condition for a basis.

2) To check the second condition, let

$B_1 = S_1 \cap \ldots\ldots \cap S_m$ and $B_2 = S'_1 \cap \ldots\ldots \cap S'_n$ be two elements of $\mathfrak{B}$. Their intersection and $B_1 \cap B_2 = (S_1 \cap \ldots\ldots \cap S_m) \cap (S'_1 \cap \ldots\ldots \cap S'_n)$ is also a finite intersection of elements of S, so it belongs to $\mathfrak{B}$.

3.3. THE ORDER TOPOLOGY

3.3.1. ORDER TOPOLOGY DEFINITION:

- If X is a simply ordered set, there is a standard topology for X, defined using the order relation. It is called the order topology. We consider it and study some of its properties.

Result:

1) Suppose that X is a set having a simple order relation <. Given elements a and b of X such that a < b, there are four subsets of X that are called the **intervals** deter- mined by a and b. They are the following:

   (a, b) = {x / a < x < b},

   (a, b] = {x / a < x ≤ b},

   [a,b) = {x / a ≤ x < b},

   [a, b] = (x / a ≤ x ≤ b}.

   The notation used here is familiar to you already in the case where X is the real line, but these are intervals in an arbitrary ordered set.

2) A set of the first type is called an **open interval** in X, a set of the last type is called a **closed interval** in X, and sets of the second and third types are called **half-open intervals.** The use of the term "open" in this connection suggests that open intervals in X should turn out to be open sets when we put a topology on X.

- Let X be a set with a simple order relation; assume X has more than one element. Let $\mathfrak{B}$ be the collection of all sets of the following types:
  1) All open intervals (a, b) in X.
  2) All intervals of the form $[a_o, b)$, where a_o is the smallest element (if any) of X.
  3) All intervals of the form $(a, b_o]$, where b_o is the largest element (if any) of X.
- The collection $\mathfrak{B}$ is a basis for a topology on X, which is called the **order topology.**

Note:

- If X has no smallest element, there are no sets of type (2), and if X has no largest element, there are no sets of type (3).

Example 1.

- The standard topology on $\mathbb{R}$, as defined in the preceding section, is just the order topology derived from the usual order on $\mathbb{R}$.

Example 2.

- Consider the set $\mathbb{R} \times \mathbb{R}$, in the dictionary order; we shall denote the general element of $\mathbb{R} \times \mathbb{R}$ by $x \times y$, to avoid difficulty with notation. The set $\mathbb{R} \times \mathbb{R}$ has neither a largest nor a smallest element, so the order topology on $\mathbb{R} \times \mathbb{R}$ has as basis the collection of all open intervals of the form $(a \times b, c \times d)$ for $a < c$, and for $a = c$ and $b < d$.
- The sub collection consisting of only intervals of the second type is also a basis for the order topology on $\mathbb{R} \times \mathbb{R}$, as you can check.

Example 3.

- The positive integers Z_+ form an ordered set with a smallest element. The order topology on Z_+ is the discrete topology, for every one-point set is open: If $n > 1$, then the one-point set $\{n\} = (n - 1, n + 1)$ is a basis element; and if $n = 1$, the one-point set $\{1\} = [1, 2)$ is a basis element.

Example 4.

- The set $X = \{1, 2\} \times Z_+$ in the dictionary order is another example of an ordered set with a smallest element. Denoting $1 \times n$ by a_n and $2 \times n$ by b_n, we can represent X by

$$a_1, a_2, \ldots ; b_1, b_2, \ldots.$$

- The order topology on X is not the discrete topology. Most one-point sets are open, but there is an exception-the one-point set $\{b_1\}$. Any open set containing b_1 must contain a basis element about b_1 (by definition), and any basis element containing b_1 contains points of the a_i sequence.

3.3.2. RAYS DEFINITION

- If X is an ordered set, and a is an element of X, there are four subsets of X that are called the rays determined by a. They are the following:

$$(a, +\infty) = \{ x / x > a\},$$

$(-\infty, a) = \{ x / x < a \}$,

$[a, +\infty) = \{ x / x \geq a\}$,

$(-\infty, a] = \{ x / x \leq a\}$.

- Sets of the first two types are called **open rays,** and sets of the last two types are called **closed rays.**

Result:

1) The use of the term "open" suggests that open rays in X are open sets in the order topology. And so they are. Consider, for example, the ray $(a,+\infty)$. If X has a largest element b_0, then $(a, +\infty)$ equals the basis element $(a, b_0]$. If X has no largest element, then $(a, +\infty)$ equals the union of all basis elements of the form (a, x), for $x > a$. In either case, $(a, +\infty)$ is open. A similar argument applies to the ray $(-\infty, a)$.
2) The open rays, in fact, form a subbasis for the order topology on X, as we now show. Because the open rays are open in the order topology, the topology they generate is contained in the order topology. On the other hand, every basis element for the order topology equals a finite intersection of open rays; the interval (a, b) equals the intersection of $(-\infty, b)$ and $(a, +\infty)$, while $[a_0, b)$ and $(a, b_0]$, if they exist, are themselves open rays.
3) Hence the topology generated by the open rays contains the order topology.

3.4. THE PRODUCT TOPOLOGY ON X ×Y

- If X and Y are topological spaces, there is a standard way of defining a topology on the Cartesian product $X \times Y$. We consider this topology now and study some of its properties.

3.4.1. PRODUCT TOPOLOGY DEFINITION:

- Let X and Y be topological spaces. The **product topology** on $X\times Y$ is the topology having as basis the collection $\mathfrak{B}$ of all sets of the form $U\times V$, where U is an open subset of X and V is an open subset of Y.
- Let us check that $\mathfrak{B}$ is a basis. The first condition is trivial, since $X \times Y$ is itself a basis element. The second condition is almost as easy, since the intersection of any two basis elements $U_1\times V_1$ and $U_2\times V_2$ is another basis element. For and the latter set is a basis element because $U_1 \cap U_2$ and $V_1\cap V_2$ are open in X and Y, respectively.

$$(U_1\times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1\cap V_2),$$

- Note that the collection $\mathfrak{B}$ is not a topology on $X \times Y$. The union of the two rectangles is not a product of two sets, so it cannot belong to $\mathfrak{B}$; however, it is open in $X \times Y$.

3.4.2. THEOREM

- If $\mathfrak{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y, then the collection $\mathcal{D} = (B \times C \,/\, B \in \mathfrak{B}$ and $C \in \mathcal{C}\}$ is a basis for the topology of $X \times Y$.

Proof:

- We apply Lemma 3.2.3. Given an open set W of $X \times Y$ and a point $x \times y$ of W, by definition of the product topology there is a basis element $U \times V$ such that $x \times y \in U \times V \subset W$.
- Because $\mathfrak{B}$ and e are bases for X and Y, respectively, we can choose an element B of $\mathfrak{B}$ such that $x \in B \subset U$, and an element C of $\mathcal{C}$ such that $y \in C \subset V$. Then $x \times y \in B \times C \subset W$. Thus the collection $\mathcal{D}$ meets the criterion of Lemma 3.2.3, so $\mathcal{D}$ is a basis for $X \times Y$.

Example 1.

- We have a standard topology on $\mathbb{R}$ the order topology. The product of this topology with itself is called the **standard topology** on $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$. It has as basis the collection of all products of open sets of $\mathbb{R}$, but the theorem just proved tells us that the much smaller collection of all products $(a, b) \times (c, d)$ of open intervals in $\mathbb{R}$ will also serve as a basis for the topology of $\mathbb{R}^2$. Each such set can be pictured as the interior of a rectangle in $\mathbb{R}^2$. Thus the standard topology on $\mathbb{R}^2$ is just the one we considered in Example 2 of section 3.2.
- It is sometimes useful to express the product topology in terms of a subbasis. To do this, we first define certain functions called projections.

3.4.3. PROJECTIONS DEFINITION

- Let $\pi_1 : X \times Y \to X$ be defined by the equation

$$\pi_1(x, y) = x;$$

- let $\pi_2 : X \times Y \to Y$ be defined by the equation

$$\pi_2(x, y) = y.$$

- The maps π_1 and π_2 are called the **projections** of $X \times Y$ onto its first and second factors, respectively.

Result:

1) We use the word "onto" because π_1 and π_2 are surjective (unless one of the spaces X or Y happens to be empty, in which case X×Y is empty and our whole discussion is empty as well!).
2) If U is an open subset of X, then the set π_1^{-1} (U) is precisely the set U×Y, which is open in X × Y.
3) Similarly, if V is open in Y, then π_2^{-1} (V) =X × V, which is also open in X × Y. The intersection of these two sets is the set U×V.

3.4.4. THEOREM

- The collection S $= \{\pi_1^{-1}(U) \,/\, U$ open in X$\} \cup \{\pi_2^{-1}(V) \,/ V$ open in Y$\}$ is a subbasis for the product topology on X × Y.

Proof:

- Let $\mathcal{T}$ denote the product topology on X ×Y; let $\mathcal{T}'$ be the topology generated by S. Because every element of S belongs to $\mathcal{T}$, so do arbitrary unions of finite intersections of elements of S. Thus $\mathcal{T}' \subset \mathcal{T}$. On the other hand, every basis element U × V for the topology $\mathcal{T}$ is a finite intersection of elements of S, since

$$U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V).$$

- Therefore, U × V belongs to $\mathcal{T}'$, so that $\mathcal{T} \subset \mathcal{T}'$ as well.

3.5.THE SUBSPACE TOPOLOGY

3.5.1. SUBSPACE TOPOLOGY DEFINITION:

- Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection $\mathcal{T}_y = \{Y \cap U \,/\, U \in \mathcal{T}\}$ is a topology on Y, called the **subspace topology.** With this topology, Y is called a subspace of X; its open sets consist of all intersections of open sets of X with Y.

RESULT:

1) It is easy to see that $\mathcal{T}_y$ is a topology. It contains ∅ and Y because $\emptyset = Y \cap \emptyset$ and $Y = Y \cap X$, where ∅ and X are elements of $\mathcal{T}$.
2) The fact that it is closed under finite intersections and arbitrary unions follows from the equations $(U_1 \cap Y) \cap (U_2 \cap Y) \ldots \ldots (U_n \cap Y) = (U_1 \cap U_2 \ldots \ldots \cap U_n) \cap Y$

$\bigcup_{\alpha \in J}(U_\alpha \cap Y) = (\bigcup_{\alpha \in J} U_\alpha) \cap Y$

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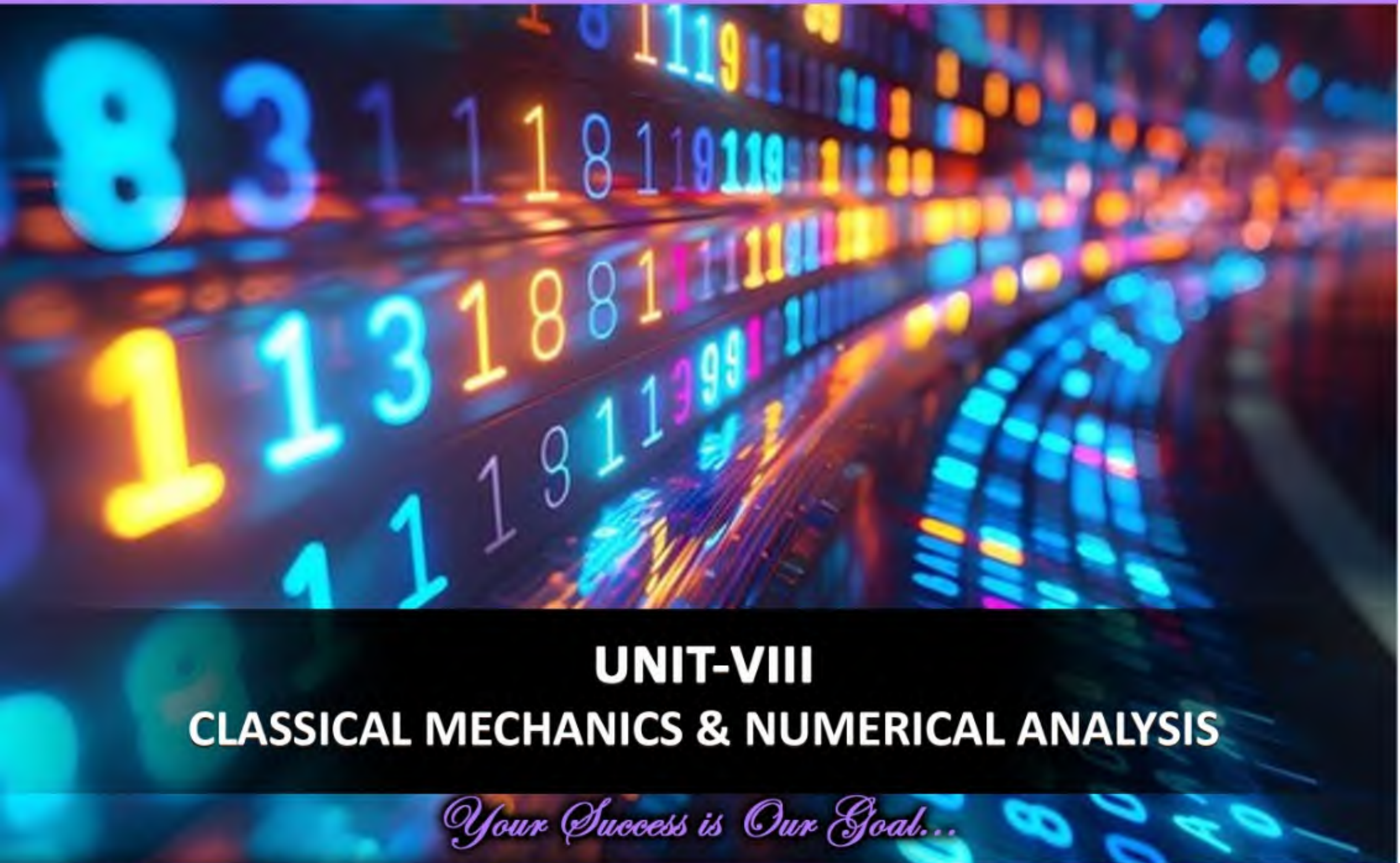

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8.2. NUMERICAL ANALYSIS		

UNIT – VIII CLASSICAL MECHANICS & NUMERICAL ANALYSIS

SYLLABUS

Unit VIII **CLASSICAL MECHANICS AND NUMERICAL ANALYSIS**

Classical Mechanics

Generalised Co-ordinates – Lagrange's equations – Hamilton's Canonical equations – Hamilton's principle – Principle of least action – Canonical transformations – Differential forms and Generating functions – Lagrange and Poisson brackets.

Numerical Analysis

Numerical solutions of algebraic and transcendental equations – Method of iteration – Newton Raphson method – Rate of convergence – Solution of Linear algebraic equations using Gauss elimination and Gauss – Seidel methods.

Finite differences – Lagrange, Hermite and Spline Interpolation, Numerical differentiation and integration – Numerical solutions of Ordinary differential equations using Picard, Euler, Modified Euler and Runge- Kutta methods.

TCA

Refference Books :

1) Classical Dynamics by Donald T.Greenwood
2) Classical Mechanics by Dr.J.C Upadhaya
3) Classical Mechanics by Hebert Goldstein, Charles P.Poole, John Safko 3rd Edition Pearson Publication
4) Classical Mechanics by R.Douglas Gregory
5) Principles of Mechanics by John L.Synge, Byron A.Griffith 2nd Edition, New York.

More Reference:

- PGTRB Previous Year Question Papers
- UGC NET Previous Year Question Papers
- SET (State eligibility Test) Previous Year Question Papers (Tamilnadu, Madhya Pradesh, Kerala, Gujarat, Chhattisgarh, Andhra Pradesh, Himachal Pradesh, West Bengal, Uttarakhand, Rajasthan, Maharashtra)

8.1.1 SOME BASIC CONCEEPT IN MECHANICS

SPACE

- Space is a geometric region occupied by bodies whose positions are described with the help of co-ordinate system.

TIME

- It is measure of the succession of the events.

MASS

- It is used to measure the resistance to change the state of rest or motion is called inertia.

FORCE

- It is the effort required to change or tends to change the state of rest.

MECHANICS AND CLASSIFICATION

- Mechanics is the science, that describes and the predicts the effect of objects either at rest or in motion.

MECHANICS

- Mechanics of Solids
  - Rigid bodies
    1. Statics (Non-Movable)
    2. Dynamics (Movable)
  - Deformable bodies
    - rength of materials
- Mechanics of fluids (Liquid or Gas)
  1. Ideal fluids.
  2. Viscous fluids.
  3. Compressable fluids.

DYNAMICS

- Dynamics is the study of the motion interacting bodies. It describes this motion in terms of postulated laws.

CLASSICAL DYNAMICS

- Dynamics is the study of the motion of interacting bodies.
- Classical dynamics is restricted to those systems of interacting bodies for which quantum mechanical effects are negligible.
- The non-relativistic theories and methods of men such as Newton, Euler, Lagrange, and Hamilton are included, as well as the more recent relativistic dynamics of Einstein.

THE MECHANICAL SYSTEM:

Definition:

- Let us consider a mechanical system consisting of N particles, where a particle is an idealized material body having its mass concentrated at a point.
- The motion of a particle is, therefore, the motion of a point in space.
- Since a point has no geometrical dimensions, we cannot specify the orientation of a particle, nor can we associate any particular rotational motion with it. In this non-relativistic treatment, we shall assume that the mass of each particle remains constant.

Equation of Motion:

- The differential equations of motion for a system of N particles can be obtained by applying Newton's laws of motion to the particles individually. For a single particle of mass m which is subject to a force$\vec{F}$, we obtain from Newton's second law the vector equation.
- For a single particle of mass m which is subject to a force

$$\vec{F} = \mathrm{m}\vec{a}$$

$$= m\frac{d\vec{v}}{dt}[\text{ where } \vec{a} = \frac{d\vec{v}}{dt}]$$

$$= \frac{d}{dt}(m\vec{v})$$

$$= \frac{d\vec{v}}{dt}[\vec{p} = m\vec{v}]((\text{since } \vec{p} \text{ is the linear momentum})$$

$$\vec{F} = \dot{\vec{p}}$$

$$\vec{F} = \mathrm{m}\vec{a}$$

$$= \mathrm{m}\frac{d\vec{v}}{dt}$$

$$= \mathrm{m}\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = m\frac{d^2\vec{r}}{dt^2}$$

$$\vec{F} = \mathrm{m}\ddot{\vec{r}}$$

- Here, $\vec{a}(\dot{\vec{v}})$ is the acceleration measured relative to an inertial frame of reference.

DEGREES OF FREEDOM

- The number of degrees of freedom =The number of coordinates- The number of independent equation of constraint.

EXAMPLE 1 :

- If the configuration of a system of N particles is described using 3N cartesian coordinates and if there are l independent equations of the constraints relating these coordinates then there are $(3N - l)$ degrees of freedom.

EXAMPLE 2 :

- The configuration of the system is specified by giving the locations of the three particles that is 9 cartesian coordinates. But reach rigid rod is represented mathematically by an independent equation of constraint. So $3N - l = 3 \times 3 - 3 = 6$ and the system has 6 a degree of freedom.

EXAMPLE 3 :

- When a particle is moving in a plane, we may describe its motion in Cartesian coordinates or, more conveniently in polar coordinates r, θ we may then

$$q_1 = x, q_2 = y$$

$$q_1 = r = (x^2 + y^2)^{\frac{1}{2}}, \qquad q_2 = \theta = tan^{-1}\left(\frac{y}{x}\right)$$

Alternatively we may also written as $x = q_1 cosq_2 y = q_1 sinq_2$

Or $x_i = x_i(q_1, q_2)$

- If the problem involves spherical symmetry we may conveniently use spherical polar co ordinates (r, θ, φ) as (q_1, q_2, q_2) such that $x = rsin\theta cos\varphi$

$$y = rsin\theta sin\varphi$$

$$z = rcos\theta$$

$$\text{Or } x = q_1 sinq_2 cosq_3$$

$$y = q_1 sinq_2 cosq_3$$

$$x = q_1 cosq_2$$

Or

$$x_i = x_i(q_1, q_2, q_3)$$

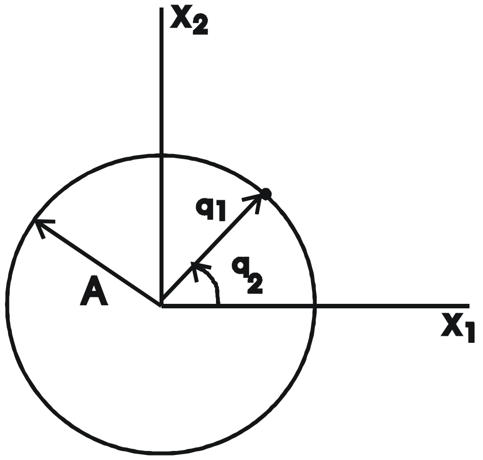
- In general the relation between the old Cartesian co-ordinates and the generalized co-ordinate can be written in the form$x_i = x_i(q_1, q_2, q_3, \ldots\ldots q_n)$.

EXAMPLE 4 :

- A rod of mass m and length l which is free to move in any direction on a frictionless inclined plane has only perpendicular to degrees of freedom as it can slide down and vertically.

EXAMPLE 5 :

- A transformation from Cartesian to generalized coordinates, consider a particle which is constrained to move on a fixed circular path of radius a , as $({x_1}^2 + {x_2}^2)^{\frac{1}{2}} = a$



- Let a single generalized coordinates q_1 represent the one degree of freedom. This polar angle can vary freely without violating the constant. In accordance with above diagram.Let us define a second generalized coordinates q_2 which is constant.

$$q_2 = a$$

- The transformation equation are

$$x_1 = q_2 \cos q_1, x_2 = q_2 \sin q_1$$

- The jacobian for this transformation is $\frac{\partial(x_1,x_2)}{\partial(q_1,q_2)} = \begin{vmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_1}{\partial q_2} \\ \frac{\partial x_2}{\partial q_1} & \frac{\partial x_2}{\partial q_2} \end{vmatrix} = -q_2$

- Hence the q's may be expressed as a functions of the $x's$ except when the Jacobian is 0 at $q_2 = 0$ in this case the radius of circle is zero and the angle q_1 is undefined.These transformation equations are

$$\boxed{q_1 = \tan^{-1}\left(\frac{x_2}{x_1}\right),\ q_2 = ({x_1}^2 + {x_2}^2)^{\frac{1}{2}},}$$

- Where we arbitrary take $0 \leq q_1 \leq 2\pi and \quad 0 < q_2 < \infty$ in the order that q's will be single-valued functions of the $x's$ These transformation equations apply at the all points on the finite x_1, x_2 plane except at the origin.

CLASSIFICATION OF DYNAMICAL SYSTEM

i.) Scleronomic

- A configuration of the system is given when the values of the generalized co ordinates $q_1, q_2, q_3, \ldots\ldots q_n$ are given by the equation of the form $x_i = x_i(q_1, q_2, q_3, \ldots\ldots q_n)$
- Which is do not depend explicitly on time.

ii.) Rheonomic system

- Rheonomic system, it is necessary to specify the time t as well then the equations will be of the form $x_i = x_i(q_1, q_2, q_3, \ldots\ldots q_n, t)$(depending on time)

REMARK

- Thus, in Scleronomic system, there will be only fixed constraints, whereas in a Rheonomic system there will be moving constraints

iii.) Conservative and non Conservative

- In a conservative system the generalized forces are derivable from a potential energy $v = v(q_1, q_2, q_3, \ldots\ldots q_n)$

  (i.e) if $Q_1, Q_2, Q_3, \ldots\ldots Q_n$ represents generalized forces then

$$Q_1 = \frac{-\partial v}{\partial q_1}, Q_2 = \frac{-\partial v}{\partial q_2}, \ldots\ldots\ldots. Q_n = \frac{-\partial v}{\partial q_n}$$

$$Q = -\nabla\vec{V}$$

- Otherwise system is non conservative

iv.) Holonomic and non-holonomic

- Let $q_1, q_2, q_3, \ldots\ldots q_n$ denote the generalized co-ordinates, describing a system and let t denotes the time and if all the constraints of the system can be expressed in equation of the form.

$$f_i(q_1, q_2, q_3, \ldots\ldots q_n, t) = 0$$

- Which is independent of velocity of $\dot{q}_1, \dot{q}_2, \dot{q}_3, \ldots\ldots \dot{q}_n$
- Where$i$ is the i^{th}component, then the system is called Holonomic,
- However if the constraint cannot be expressed as relation among the generalized coordinates, then the system is called non-holonomic.

v.) Bilateral and unilateral system

- If at any point on constraint surface, both the forward and backward motions are possible,such a system is called Bilateral
- In this system, the constraint relation in the form of equation but not in the form of inequalities.
- However, if at some point on the constraint surface, the forward motion is not possible, while the constraint relation are expressed in the form of inequalities,then such a system is called Unilateral

vi.) Simple system

- Systems which are scleronomic conservative holonomic are called simple system. It is also known as general system.

Key points

Scleronomic- Time independent

Rheonomic system- Time dependent

Holonomic- Velocity independent

Non-Holonomic- Velocity dependent

Conservative - Generalized force derived form

Non Conservative - potential energy, otherwise it is non conservative

Bilateral -if forward and backward motion is possible

- Constraint relation are in the form of equation

Unilateral - if forward and backward motion is NOT possible, only backward motion is possible

- Constraint relation are in the form of inequalities

Virtual work

- Suppose the configuration of a system of N particles is given by 3N Cartesian coordinates $x_1, x_2, \dots\dots x_{3N}$ which are measured relative to an inertial frame and may be subject to constraints. At any given time, let assume that the coordinates moves through infinite decimal displacement$\delta x_1, \delta x_2, \delta x_3, \dots.\delta x_{3N}$ which are virtual or imaginary in the sense that they are assumed to occur without the passage of time, and do not necessarily conform to constraints. This small change the δx in the configuration of the system is known as virtual displacements.

$$\delta w = \sum_{j=1}^{3N} F_j \delta_{x_j} = \sum_{i=1}^{N} F_I \delta_{r_i}$$

$$\delta w_c = \sum_{i=1}^{N} R_i \delta_{r_i} = 0$$

$$\delta w_c = R_1 \delta_{r_1} + R_2 \delta_{r_2} = 0$$

Principal of virtual work

- We can imagine arbitrary instantaneous change in the position vectors of the particles of the system, virtual displacements. An infinitesimal virtual displacement of i^{th} particle of a system of N particles is denoted by δr_i. This is the displacement of position coordinates only and does not involve variation of time.

  i.e.,$\delta r_i = \delta r_i(q_1, q_2, \dots \dots q_n)$

- Suppose the system is in equilibrium, then the total force on any particle is zero

  i.e.,$F_i = 0 \quad i = 1,2, \dots \dots., N$

- The virtual work of the force F_i, in the virtual displacement δr_i will also be zero

  ie.,$\delta W_i = F_i.\delta r_i = 0$

- Similarly, the sum of virtual work for all the particles must vanish

  i.e.,$\delta W = \sum_{i=1}^{N} F_i.\delta r_i = 0$

- This result represents the principle of virtual work which states that,

The work done is zero in the case of an arbitrary virtual displacement of a system from a position of equilibrium

$F_i + R_i = 0$ ($F_i$is Applied force R_iConstrain Force)

$$\sum_{i=1}^{N}(F_i + R_i).\delta r_i = \sum_{i=1}^{N} F_i.\delta r_i + \sum_{i=1}^{N} R_i.\delta r_i$$

- We restrict ourselves to the systems where the virtual work of the forces of constraints is zero e.g., in case of a rigid body.
- Then $\sum_{i=1}^{N} F_i.\delta r_i = 0$
- i.e., for equilibrium of a system, the virtual work of applied forces is zero. We see that the principle of virtual work deals with the statics of a system of particles. However, we want a principle to deal with the general motion of the system and such a principle was developed by D' Alembert.

D'ALEMBERT PRINCIPLE

- According to Newton's second law of motion, the force acting on the i^th particle is given by $F_i - \dot{p}_i = 0 \quad i = 1,2,\dots..N$
- These equations mean that any particle in the system is in equilibrium under a force, which is equal to the actual force F_i, plus a reversed effective force $\dot{p}_i$. Therefore, for virtual displacement δr_i,

$$\sum_{i=1}^{N}(F_i - \dot{p}_i).\delta r_i = 0$$

- But $F_i = F_i{}^a + f_i = 0$
- Then

$$\sum_{i=1}^{N}(F_i{}^a - \dot{p}_i).\delta r_i + \sum_{i=1}^{N} f_i.\delta r_i = 0$$

- Again, we restrict ourselves to the systems for which the virtual work of the constraints is zero

ie,$\sum_i f_i \,.\, \delta r_i = 0$ Then$\sum_{i=1}^{N}(F_i{}^a - \dot{p}_i).\delta r_i = 0.$

A workless constraint is any bilateral constraint such that the virtual work of the corresponding constraint forces is zero for any virtual displacement which is consistent with the constraints.

SUMMARIZED PRINCIPLE OF VIRTUAL WORK

The necessary and sufficient condition for the static equilibrium of any initially motionless Scleronomic system which is subject to workless constraints is that zero virtual work by done by the applied forces in the moving through an arbitrary virtual displacement satisfying the constraints.

D'Alembert's principal

- Consider a system of N particles and write the equation of motion for each particle in the form $F_i + R_i - m_i\ddot{r}_i = 0$ (F_i is Applied force R_i is Constrain Force)

LAGRANGE'S FORM OFD' ALEMBERT'S PRINCIPAL

$$\delta W = \sum_{i=1}^{n} (F_i + R_i - m_i\ddot{r}_i)\delta r_i = 0$$

Example

- A particle of mass m is suspended by a massles wire of length $r = a + bcos\omega t$ $(a > b > 0)$to form a spherical pendulum. Equation of motion is $(a + bcos\omega t)\ddot{\phi}sin\theta - 2bw\dot{\phi}sin\omega t sin\theta + 2(a + bcos\omega t)\dot{\theta}\phi cos\theta = 0$

The necessary and sufficient condition for static equilibrium is that all Q'S due to the applied force is zero.

ENERGY AND MOMENTUM

POTENTIAL ENERGY

- Let us consider a single particle whose position given by the Cartesian coordinates(x,y,z).suppose that the total force F acting on the particle has the components

$$F_x = -\frac{\partial V}{\partial x}$$

$$F_y = -\frac{\partial V}{\partial y}$$

$$F_z = -\frac{\partial V}{\partial z}$$

- Where the potential energy function $V(x, y, z)$ is single-valued function of position only, that is not a function of velocity or time.A force F meeting these conditions is known as a conservative force.

- Let us consider the work dW done by the force F as it moves through an infinitesimal displacement dr. we have $dW = F.dr = F_x dx + F_y dy + F_z dz$

$$W = \int_A^B F.dr = -\int_A^B dV = V_A - V_B$$

Work done is in any closed path zero (ie)$\oint \mathrm{F.dr} = 0$

=Two forces systems according on a given rigid body or EQUIPOLLENT if they have the same total force and the same total moment with respect to an arbitrary point

WORK AND KINETIC ENERGY

- Suppose we define the kinetic energy T of a particle of mass m by

$$T = \frac{1}{2}mv^2$$

- Where v is the velocity of the particle relative to an inertia reference frame. Let us considered the line integral of equation

$$W = \int_A^B F.dr = -\int_A^B dV = V_A - V_B$$

- which gives the work done on the particle by the total force F of as the particle moves over a certain path from A to B. In accordance with Newton's law of motion we cancan replace F by $m\ddot{r}$ and obtain

$$W = m\int_A^B \ddot{r}.dr = \frac{1}{2}m\int_A^B \frac{d}{dt}(\dot{r}\,.\dot{r})dt = \frac{1}{2}m\int_A^B d(v^2)$$

- Where each integral is evaluated over the same path. then, using the definition of kinetic energy, we obtain $W = \frac{1}{2}m({v_B}^2 - {v_B}^2) = T_B - T_A$ this is called principle of work and **kinetic energy:**

The increase in kinetic energy of a particle as it moves from one orbitary point to another is equal to the work done by the forces acting on the particle during the given interval.

- Note that the force F may arise from any source; it need not be conservative. Furthermore, force components which remain normal to the particle velocity v do no work and can be neglected the applying the principle.

CONSERVATION OF ENERGY.

- If the only forces acting on a given particle are conservative then

  $W = \int_A^B F.dr = -\int_A^B dV = V_A - V_B$ applies and, with the aid of

- $W = \frac{1}{2}m({v_B}^2 - {v_B}^2) = T_B - T_A$we obtain

  $V_A - V_B = T_B - T_A$ (or) $V_A + T_A = V_B + T_B$= E

- Since the points A and B arbitrary, we conclude that the total mechanical energy E remains constant during the motion of the particle. This is the principle of conservation energy.
- Now let us consider the more general case of a system of N particles whose configurations is satisfied by the Cartesian $x_1, x_2, \ldots \ldots \ldots .. x_{3N}$. If the only force which do work on the system during its motion are given by
- $F_j = -\frac{\partial V}{\partial x_j}$
- Where the potential energy$V(x_1, x_2, \ldots \ldots \ldots .. x_{3N})$ is a single-valued function of position only, then the total energy E is again conserved.
- Frequently it is convenient to specify the configuration of a system of particles by using generalized coordinates.

EQUILIBRIUM AND STABILITY

- Consider a system of N particles whose applied forces are conservative and are obtained from a potential energy function of the form V$(x_1, x_2, \ldots \ldots \ldots .. x_{3N})$.
- The virtual work of these applied forces $\delta W = -\sum_{j=1}^{3N} \frac{\partial V}{\partial x_j} \delta x_j = -\delta V$
- Which we note is linear in the $\delta x's$and is, therefore, the first variation of the potential energy. Then, using the principle of virtual work,

The necessary and sufficient condition for the static equilibrium of this system is that $\delta V = 0$

- A holonomic system having independent q's the condition that $\delta V = 0$

KINETIC ENERGY OF A SYSTEM

$$T = \frac{1}{2}m\dot{r}_c^{\ 2} + \frac{1}{2}\sum_{i=1}^{N} m\dot{\rho}_c^{\ 2}$$

KONIG'S THEOREM

- The total kinetic energy of a system is equal to sum of $\frac{1}{2}m\dot{r}_c^{\ 2}$ the kinetic energy due to a particle having a mass equal to the total mass of the system and moving with the velocity of the centre of mass and $\frac{1}{2}\sum_{i=1}^{N} m\dot{\rho}_c^{\ 2}$the kinetic energy due to the motion of the system relative to its centre of mass.

THE GENERALIZED MOMENTUM

- Consider a system whose configuration is described by n generalized coordinates. Let us define the lagrangian function $L(q, \dot{q}, t)$

$$L = T - V$$

- The generalize the momentum $p_i$associated with the generalized coordinate q_i is defined by the equation $p_i = \frac{\partial T}{\partial \dot{q}_i}$

<u>EXAMPLE 1 :</u>

- Consider a free particle of mass m whose posititonis given by the Cartesian coordinates(x, y, z) the kinetic energy is $T = \frac{m}{2}\ (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

<u>EXAMPLE 2 :</u>

- If the position of the particle is given by the spherical coordinates (r, θ, ϕ) the kinetic energy is T=$\frac{m}{2}\ (\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2 sin^2\phi)$ using this equation $p_i = \frac{\partial T}{\partial \dot{q}_i}$ we get

$$p_r = m\dot{r}$$

$$p_\theta = mr^2\dot{\theta}$$

$$p_\phi = mr^2\dot{\phi}sin^2\phi$$

- Where $p_r$is the linear momentum in the radial direction
- $p_\theta$is the horizontal component of the angular momentum
- $p_\phi$is the vertical component of angular Momentum

EXERCISE : 1

1. For a single of mass m which is subject to a force F then

(a) $F = ma$

(b) $F = \frac{dp}{dt}$

(c) (c) $F = mv$

(d) (d) $F = \frac{ma}{2}$

2. The equation of a motion for the system of N particle is

(a) $F_i + R_i + m_i\ddot{r}_i = 0$

(b) $\ddot{r}_i = F_i + R_i + m_i$

(c) $F_i + R_i + m_i\ddot{r}_i = 0$

(d) $F_i + R_i - m_i\ddot{r}_i = 0$

3. The equation of a motion for the system of N particle is $\boldsymbol{F_i + R_i - m_i\ddot{r}_i = 0}$ then $\boldsymbol{F_i, R_i}$is

(a) Constraint and applied force

(b) both Constraint force

(c) applied and constraint force

(d) both applied force

4. The number of degrees of freedom is

(a) The number of coordinates –number of independent of equation of constraint.

(b) Number of independent of equation of constraint -The number of coordinates.

(c) The number of coordinates + number of independent of equation of constraint.

(d) Number of independent of equation of constraint -The number of coordinates

5. If the configuration of a system of N particle is described using 3N Cartesian coordinates and if there are *l* independent equation of constraints relating these coordinates then the degrees of freedom is

(a) $3N - l$

(b) $3N - 3l$

(c) $N - 3l$

(d) $N - l$

6. Three particles are connected by rigid rod to form a triangle body eith particle at its corners. Then the number of degrees of freedom of a system is (*POLY TRB 2021*)

(a) 4

(b) 3

(c) 6

(d) 9

7. Which one is noholonomic constraints

(a) $\sum_{i=1}^{n} a_{ji}dq_i - a_{ji}dt = 0 \quad (j = 1,2, \dots \dots .. m)$

(b) $\sum_{i=1}^{n} a_{ji}dq_i + dt = 0 \quad (j = 1,2, \dots \dots .. m)$

(c) $\sum_{i=1}^{n} a_{ji} + a_{ji}dt = 0 \quad (j = 1,2, \dots \dots .. m)$

(d) $\sum_{i=1}^{n} a_{ji}dq_i + a_{ji}dt = 0 \quad (j = 1,2, \dots \dots .. m)$

8. Which axis the virtual work of the constrain force is

(a) $\delta W_c = \sum_{i=1}^{n} R_i \,.\, \delta r_i$

(b) $\delta W_c = \sum_{i=1}^{n} R_i + \delta r_i$

(c) $\delta W_c = \sum_{i=1}^{n} R_i - \delta r_i$

(d) $\delta W_c = \sum_{i=1}^{n} R_i$

9. The necessary and sufficient condition for static equilibrium is that all the Q's due to the applied force to_______________

(a) 1

(b) -1

(c) Infinity

(d) 0

10. The work done in moving around any closed path is

(a) $\oint F.dr \neq 0$

(b) $\oint F.dr = 0$

(c) $\oint F.dr = m.r_i$

(d) $\oint F.dr = \frac{1}{2} m r^2$

TCA

SCAN QR CODE FOR ANSWERS

8.1.2 LAGRANGE'S EQUATIONS

Lagrange's equation of d' Alembert principle

- Consider a system of N particles. The transformation equations for the position vectors of the particles are
- $r_i = r_i(q_1, q_2, \dots \dots . q_n, t)\ \ i = 1,2, \dots . . N$ …………(1)
- Where t is time and $q_k (k = 1.,2, \dots \dots . n)$ are generalized coordinates.
- Differentiating equation(1) with respect to t, we obtain the velocity of the i th particle

$$\text{i.e.,} \frac{dr_i}{dt} = \frac{\partial r_i}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial r_i}{\partial q_2}\frac{dq_2}{dt} + \cdots + \frac{\partial r_i}{\partial q_k}\frac{da_k}{dt} + \cdots + \frac{\partial r_i}{\partial q_n}\frac{dq_n}{dt} + \frac{\partial r_i}{\partial t}$$

$$\text{or} v_i = \dot{r_i} = \sum_{k=1}^{n} \frac{\partial r_i}{\partial q_1}\dot{q_k} + \frac{\partial r_i}{\partial t}$$

- where$\dot{q_k}$ are the generalized velocities.
- The virtual displacement is given by

$$\delta r_i = \frac{\partial r_i}{\partial q_1}\delta q_1 + \frac{\partial r_i}{\partial q_2}\delta q_2 + \cdots + \frac{\partial r_i}{\partial q_k}\delta q_k + \cdots + \frac{\partial r_i}{\partial q_n}\delta q_n$$

$$\text{Or} \qquad \delta r_i = \sum_{k=1}^{n} \frac{\partial r_i}{\partial q_k}\delta q_k$$

- Since by the definition the virtual displacement do not dependent time,
- According D' Alembert principle,

$$\sum_{i=1}^{n} (F_i - \dot{p_i}).\delta r_i = 0$$

$$\text{Here} \sum_{i=1}^{N} F_i.\delta r_i = \sum_{i=1}^{N} F_i.\sum_{k=1}^{n} \frac{\partial r_i}{\partial q_k}\delta q_k = \sum_{k=1}^{n}\sum_{i=1}^{N}[F_i.\frac{\partial r_i}{\partial q_k}]\,\delta q_k = \sum_{k=1}^{n} G_k \delta q_k$$

$$\text{Where} G_k = \sum_{i=1}^{N} F_i\,.\frac{\partial r_i}{\partial q_k}$$

$$G_k = \sum_{i-1}^{N} F_i\,.\frac{\partial r_i}{\partial q_k}$$

are called the components of generalized force associated with the generalized coordinates q_k. This may be mentioned that as the dimensions of the generalized coordinates need not be those of length; similarly the generalized force components G_K may have dimensions different than those of force. However, the dimensions of $G_k \delta q_k$, are those of work.

- For example, if δq_k, has the dimensions of length, G_k will have the dimensions of force, if δq_khas the dimensions of angle θ, G_k, will have the dimensions of torque (τ).

- Further $\sum_{i=1}^{N} \dot{p}_i \, \delta r_i = \sum_{i=1}^{N} m_i . \dot{r}_i . \sum_{k=1}^{n} \frac{\partial r_i}{\partial q_k} \delta q_k = \sum_{k=1}^{n} \left[\sum_{i=1}^{N} m_i . \ddot{r}_i . \frac{\partial r_i}{\partial q_k} \right] . \delta q_k$

  Now $\sum_{i=1}^{N} m_i . \ddot{r}_i \frac{\partial r_i}{\partial q_k} = \sum_{i=1}^{N} \left[\frac{d}{dt} \left(m_i . \dot{r}_i . \frac{\partial r_i}{\partial q_k} \right) - m_i . \dot{r}_i . \frac{d}{dt} \left(\frac{\partial r_i}{\partial q_k} \right) \right]$…………..(1)

- It is easy to prove that

$$\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_k}\right) = \frac{\partial}{\partial q_k}\left(\frac{dr_i}{dt}\right) = \frac{\partial v_i}{\partial q_k}$$

- and $$\frac{\partial r_i}{\partial q_k} = \frac{\partial v_i}{\partial \dot{q}_k}$$

Therefore equation (1) is

$$\sum_{i=1}^{N} m_i . \ddot{r}_i . \frac{\partial r_i}{\partial q_k} = \sum_{i=1}^{N} [m_i . v_i . \frac{\partial v_i}{\partial \dot{q}_k} [m . v_i . \frac{\partial v_i}{\partial \dot{q}_k}] - m_i . v_i . \frac{\partial v_i}{\partial \dot{q}_k}] . \delta q_k$$

$$\sum_{i=1}^{N} \dot{p}_i . \delta r_i = \sum_{k=1}^{n} \sum_{i=1}^{N} [\frac{d}{dt} (m_i . v_i . \frac{\partial v_i}{\partial \dot{q}_k}) - m_i . v_i . \frac{\partial v_i}{\partial \dot{q}_k}] . \delta q_k$$

$$= \sum_{i-1}^{N} \left[\frac{d}{dt} \left\{ \frac{\partial}{\partial \dot{q}_k} \left(\sum_{i=1}^{N} \frac{1}{2} m_i (v_i . v_i) \right) \right\} - \frac{\partial}{\partial q_k} \left\{ \sum_{i=1}^{N} \frac{1}{2} m_i (v_i . v_i) \right\} \right] \delta q_k$$

$$\sum_{k=1}^{n} \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} \right] \delta q_k \text{………(**)}$$

Here $\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_k}\right) = \sum_{j=1}^{n} \frac{\partial^2 r_i}{\partial q_j} \dot{q}_J + \frac{\partial r_i}{\partial t}$ ____________________(i)

- And its partial derivative with respect q_k is

$\frac{\partial v_i}{\partial q_k} = \frac{\partial}{\partial q_k}\left(\frac{dr_i}{dt}\right) = \sum_{j=1}^{n} \frac{d}{dt}\left(\frac{\partial^2 r_i}{\partial q_j}\right) - \frac{\partial T}{\partial q_k} \dot{q}_J + \frac{\partial^2 r_i}{\partial q_k \partial t}$________________________(ii)

- From equation (i) and (ii)

$$\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_k}\right) = \frac{\partial v_i}{\partial q_k}$$

$$\frac{\partial v_i}{\partial q_k} = \frac{\partial v_i}{\partial \dot{q}_k} \left[\sum_{j=1}^{n} \frac{\partial r_i}{\partial q_k} \dot{q}_J + \frac{\partial r_i}{\partial t} \right] = \frac{\partial r_i}{\partial q_j} \delta_{jk} = \frac{\partial r_i}{\partial q_k}$$

- As the constraints are holonomic and

$$\frac{\partial \dot{q}_J}{\partial \dot{q}_k} = \delta_{jk}$$

- Where $\sum_i \frac{1}{2} m_i (v_i . v_i) = \sum_i \frac{1}{2} m_i {v_i}^2 = T$ is the kinetic energy of the system.

- Substituting for $\sum_i F_{i.} \, \delta r_i$ from (**) in (*)

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MATHEMATICS

UNIT-X

PROBABILITY THEORY

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UNIT X: PROBABILITY THEORY

SYLLABUS

Sample space – Discrete Probability – Independent events – Baye's theorem – Random variables and Distribution functions (Univariate and Multivariate) – Expectation and Moments – Moment Generating function – Characteristic functions and Cumulants – Independent Random variables – Marginal and conditional distributions – Probability inequalities (Tchebyshev, Markov, Jensen) – Modes of convergence, Weak and Strong laws of large numbers – Central limit theorem (i.i.d case). Probability distributions – Binomial, Poisson, Uniform, Normal, Exponential, Gamma, Beta, Cauchy distributions – Standard Errors – Sampling distributions of t, F and Chi square and their uses in tests of significance – ANOVA – Large sample tests for mean and proportions.

REFERENCE BOOKS:

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Chapter 1

Probability and Random Variables

The theory of mathematical probability has its origin in the 17th century. There are three different approaches of measuring probabilities. They are classical probability, relative frequency of occurrence and axiomatic probability.

1.1 Preliminaries - Sample Space

Definition 1.1.1 (Trial or Random Experiment)**.** Experiment can be repeated under the same conditions but the results (Outcomes) cannot be predicted in any replication; any one of the possible way is called random experiments or trial.

For example, throwing a die is trial and tossing a coin is trial.

Definition 1.1.2 (Event)**.** Any possible outcomes of a random experiment or trial are called as event or cases.

For example, Getting 1 or 2 or $\cdots$ or 6 on a die is an event.

Definition 1.1.3 (Simple event)**.** Event consisting of only one sample point of a sample space is called a simple event.

For example, let a die be rolled once, and A is the events that face number 5 turned up, and then A is a simple event.

Definition 1.1.4 (Compound events)**.** When an event is decomposable in to a number of sample events, then it is called as compound events.

For example, the sum of two numbers shown by the upper faces of the two die is seven in the upper spaces of the two dice is seven in the simultaneous throw of the two unbiased dice, is a compound event as it can be decomposable.

Definition 1.1.5 (Sample space)**.** The set of all outcomes or events of an experiment is called as sample space. It is denoted as S.

***Example* 1.1.1:** If the outcome of an experiment consists in the determination of the sex in a newborn child, then

$$S = \{g, b\}$$

where the outcome g means that the child is a girl and b that it is a boy.

***Example* 1.1.2:** If the outcome of an experiment is the order of finish in a race among the 7 horses having post positions 1,2,3,4,5,6,7, then

$$S = \{ \text{ all } 7! \text{ permutations of } (1, 2, 3, 4, 5, 6, 7)\}.$$

The outcome $(2, 3, 1, 6, 5, 4, 7)$ means, for instance, that the number 2 horse comes in first, then the number 3 horse, then the number 1 horse , and so on.

***Example* 1.1.3:** If the experiment consists of tossing two dice, then the sampe space consists of the 36 points

$$S = \{(i, j) : i, j = 1, 2, 3, 4, 5, 6\}$$

where the outcome (i, j) is said to occur if i appears on the leftmost die and j on the other side.

***Example* 1.1.4:** If the experiment consists of measuring (in hours) the lifetime of a transitor, then the sample space consists of all nonnegative real numbers. That is,

$$S = \{x : 0 \leq x < \infty\}.$$

***Example* 1.1.5:** A sample space of tossing a coin once is

$$\mathrm{S} = \{\mathrm{H}, \mathrm{T}\}, \mathrm{n(S)} = 2.$$

***Example* 1.1.6:** A sample space of tossing a coin twice,

$$\mathrm{S} = \{\mathrm{HH}, \mathrm{HT}, \mathrm{TH}, \mathrm{TT}\}, \mathrm{n(S)} = 4$$

Definition 1.1.6 (Mutually exclusive events)**.** Events that are so related among them are said to be mutually exclusive, if the occurrence of one excludes the possibility of the occurrence of the other or in other words, two events are mutually exclusive.

For example, In tossing a coin, the events head and tail are mutually exclusive, because if the outcome is head, then the possibility of getting a tail in the same trial is ruled out,

Definition 1.1.7 (Equally likely)**.** Events said to be equally likely if there is no reason to expect any one in preference to other.

For example, in throwing a die all the six faces are equally likely to occur.

Definition 1.1.8 (Independent events)**.** A set of events are said to be independent if the occurrence of any event does not affect the chance of the occurrences of any other event of the set.

For example: When two coins (A & B) are tossed together or separately, the occurrence of the head or the tail in the case of first coin(B) does not affect the probability of the occurrence of the second $\text{coin}(B)$.

***Example* 1.1.7:** A die is rolled n times. The sample space is the pair (S, E), where S is the set of all n-tuples $(x_1, x_2, \cdots, x_n)$, $x_i \in \{1, 2, 3, 4, 5, 6\}, i = 1, 2, \cdots, n$, and E is the class of all subsets of S, which contains 6^n elementary events.

The event A that 1 shows at least once is the set

$$A = \{(x_1, x_2, \cdots, x_n) : \text{ at least one of } x_i's \text{ is } 1\} = S - \{(x_1, x_2, \cdots, x_n) : \text{ none of the } x_i's \text{ is } 1\}$$

$$A = S - \{(x_1, x_2, \cdots, x_n) : x_i \in \{2, 3, 4, 5, 6\}, i = 1, 2, \cdots, n\}.$$

***Example* 1.1.8:** Consider a pointer that is free to spin about the center of a circle. If the pointer is spun by impulse, it will finally come to rest at some point. On the assumption that the mechanism is not rigged in any manner, each point on the circumference is a possible outcome of the experiment. The set S consists of all points $0 \leq x < 2\pi r$, where r is the radius of the circle. Every one-point set $\{x\}$ is a simple event, namely, that the pointer will come to rest at x. The events of interest are those in which the pointer stops at a point belonging to a specified arc. Hence E is taken to be the Borel σ-field of subsets of $[0, 2\pi r)$.

Definition 1.1.9 (Classical definition of Probability)**.** If an experiment has ' n ' mutually exclusive, equally likely and exhaustive cases, out of which ' m ' favorable to the happening of the event A, then the probability of the happening of A is denoted by $\mathrm{P}(A)$ and is defined as $\mathrm{P}(A) = m/n$, where m is the favourable cases and where n is the total number of cases.

In general, Probability of an event = Number of favourable outcomes / Total number of outcomes.

1.1.1 Axioms of Probability

Let S be a sample space and A be an event in $S.$ then $P(A)$ is called the probability of the event if the following conditions are satisfied:

1. $\mathrm{P(A)} \geq 0$
2. $\mathrm{P(S)} = 1$
3. If A and B are mutually exclusive events. $\mathrm{P(A \cup B) = P(A) + P(B)}$

Relative frequency theory of Probability

Classical approach is useful for solving problems involving game of chances, ex: throwing dice, coins etc., but if applied to other types of problems it does not provide answers. If the experiment be repeated a large number of times under essentially identical conditions, the limiting value of the ratio of the number of times the event A happens to the total number of trials of the experiments as the number of trials increases indefinitely, is called the probability of the occurrence of A.

Thus $\mathrm{P(A)} = \lim_{n\to\infty} \frac{m}{n}$

1.1.2 Theorems of probability

1. Addition theorem
2. Multiplication theorem

Addition theorem of probability

Two events A and B are said to be mutually exclusive, if the occurrence of one event precludes the occurrence

The probability that Event A or Event B occurs is equal to the probability that Event A occurs plus the probability that Event B occurs minus the probability that both Events A and B occur.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Rule of Addition

If A and B are defined on a sample space, then: $P(A \text{ OR } B) = P(A) + P(B) - P(A \text{ AND } B)$.

If A and B are mutually exclusive, then $P(A \text{ AND } B) = 0$. Then $P(A \text{ OR } B) = P(A) + P(B) - P(A \text{ AND } B)$ becomes $P(A \text{ OR } B) = P(A) + P(B)$.

Multiplication theorem of probability

This rule states that if the two events A and B are independent the product of their separate probability gives the probability of joint occurrence

$$\mathrm{P(A \text{ and } B) = P(A) \times P(B)}$$

***Example* 1.1.9:** In the experiment of tossing two coins fin the probability of both are heads.

Multiplication rule for dependent events

This rule states if two events, A and B are dependent, the probability of occurrence of one event is dependent on the occurrence of the other event.

$$\mathrm{P(A \text{ and } B) = P(A) \times P(A/B)}$$

If A and B are two events defined on a sample space, then: $P(A \text{ AND } B) = P(B)P(A \mid B)$.

This rule may also be written as $\mathrm{P(A \mid B) = P(A}$ AND B)P(B).

$\mathrm{P(A \mid B) = P(A}$ AND B)P(B) (The probability of A given B equals the probability of A and B divided by the probability of B.)

If A and B are independent, then $P(A \mid B) = P(A)$.

Then $P(A$ AND $B) = P(A \mid B)P(B)$ becomes $P(A$ AND $B) = P(A)P(B)$.

1.2 Conditional probability

Let A and B any two independent events, the conditional probability of A and B will be represented as follows.

1. Conditional probability for A given that B has happened,

   That is, $$\mathrm{(A/B)} = \frac{\mathrm{P(A \cap B)}}{\mathrm{P(B)}}$$

2. Conditional probability for B given that A has happened,

   That is,
   $$\mathrm{(A/B)} = \frac{\mathrm{P(A \cap B)}}{\mathrm{P(A)}}$$

1.2.1 Bayes Theorem

Bayes' theorem (also known as Bayes' rule) is a useful tool for calculating conditional probabilities. Bayes' theorem can be stated as follows:

Theorem 1.2.1. Let $A_1, A_2, \ldots, A_n$ be a set of mutually exclusive events that together form the sample space S. Let B be any event from the same sample space, such that $P(B) > 0$. Then,

$$P\left(A_k \mid B\right) = \frac{P\left(A_k \cap B\right)}{\left[P\left(A_1 \cap B\right) + P\left(A_2 \cap B\right) + \ldots + P\left(A_n \cap B\right)\right]}$$

***Note* 1:** Invoking the fact that $\mathrm{P\left(A_k \cap B\right) = P\left(A_k\right) P\left(B \mid A_k\right)}$, Baye's theorem can also be expressed as

$$P\left(A_k\right) P\left(B \mid A_k\right)$$

$$P\left(A_k \mid B\right) =$$

$$\mathrm{\left[P\left(A_1\right) P\left(B \mid A_1\right) + P\left(A_2\right) P\left(B \mid A_2\right) + \ldots + P\left(A_n\right) P\left(B \mid A_n\right)\right]}$$

We can write the equation as

$$P\left(\mathrm{A}_i/\mathrm{B}\right)=\frac{P(B\mid Ai)\times P(Ai)}{\sum(i=1\text{ to }n)P(B\mid Ai)+P(Ai)}$$

***Example* 1.2.1:** What is the probability of getting a 2 or a 5 when a die is rolled?

Solution:

Taking the individual probabilities of each number, getting a 2 is $1/6$ and so is getting a 5 .

Applying the formula of compound probability,

Probability of getting a 2 or a 5 ,

$$\begin{aligned}&\mathrm{P}(\text{ 2 or }5)=\mathrm{P}(2)+\mathrm{P}(5)-\mathrm{P}(2\text{ and }5)\\ &\Rightarrow\quad 1/6+1/6-0\\ &\Rightarrow\quad 2/6=1/3.\end{aligned}$$

***Example* 1.2.2:** Consider the example of finding the probability of selecting a black card or a 6 from a deck of 52 cards.

Solution:

We need to find out $\mathrm{P}(\mathrm{B}$ or 6$)$ Probability of selecting a black card $=26/52$

Probability of selecting a $6\quad=4/52$

Probability of selecting both a black card and a $6=2/52$

$$\begin{aligned}P(B\text{ or }6)\quad&P(B)+P(6)-P(B\text{ and }6)\\&=26/52+4/52-2/52\\&=28/52\\&=7/13.\end{aligned}$$

***Example* 1.2.3:** Say, a coin is tossed twice. What is the probability of getting two consecutive tails?

Solution:

Probability of getting a tail in one toss $=1/2$

The coin is tossed twice. So $1/2*1/2=1/4$ is the answer.

Here's the verification of the above answer with the help of sample space.

When a coin is tossed twice, the sample space is $\{(\mathrm{H},\mathrm{H}),(\mathrm{H},\mathrm{T}),(\mathrm{T},\mathrm{H}),(\mathrm{T},\mathrm{T})\}$.

Our desired event is (T, T) whose occurrence is only once out of four possible outcomes and hence, our answer is $1/4$.

***Example* 1.2.4:** Consider another example where a pack contains 4 blue, 2 red and 3 black pens. If a pen is drawn at random from the pack, replaced and the process repeated 2 more times, What is the probability of drawing 2 blue pens and 1 black pen?

Solution

Here, total number of pens $= 9$

Probability of drawing 1 blue pen $= 4/9$

Probability of drawing another blue pen $= 4/9$

Probability of drawing 1 black pen $= 3/9$

Probability of drawing 2 blue pens and 1 black pen $= 4/9 * 4/9 * 3/9 = 48/729 = 16/243$

***Example* 1.2.5:** A pack contains 4 blue, 2 red and 3 black pens. If 2 pens are drawn at random from the pack, NOT replaced and then another pen is drawn. What is the probability of drawing 2 blue pens and 1 black pen?

Solution:

Probability of drawing 1 blue pen $= 4/9$

Probability of drawing another blue pen $= 3/8$

Probability of drawing 1 black pen $= 3/7$

Probability of drawing 2 blue pens and 1 black pen $= 4/9 * 3/8 * 3/7 = 1/14$

***Example* 1.2.6:** What is the probability of drawing a king and a queen consecutively from a deck of 52 cards, without replacement.

Solution:

Probability of drawing a king $= 4/52 = 1/13$.

After drawing one card, the number of cards are 51.

Probability of drawing a queen $= 4/51$.

Now, the probability of drawing a king and queen consecutively is $1/13 * 4/51 = 4/663$.

***Example* 1.2.7:** In a class, 40% of the students study math and science. 60% of the students study math. What is the probability of a student studying science given he/she is already studying math?

Solution:

$\mathrm{P}(\mathrm{M}$ and S$) = 0.40$

$P(M) = 0.60$

$\mathrm{P}(\mathrm{S} \mid \mathrm{M}) = \mathrm{P}(\mathrm{M}$ and S$)/\mathrm{P}(\mathrm{S}) = 0.40/0.60 = 2/3 = 0.67$

***Example* 1.2.8:** A single coin is tossed 5 times. What is the probability of getting at least one head?

Solution:

Consider solving this using complement.

Probability of getting no head $= \mathrm{P}($ all tails $) = 1/32$

$P($ at least one head $) = 1 - P($ all tails $) = 1 - 1/32 = 31/32$.

***Example* 1.2.9:** What is the probability of the occurrence of a number that is odd or less than 5 when a fair die is rolled?

Solution:

Let the event of the occurrence of a number that is odd be 'A' and the event of the occurrence of a number that is less than 5 be ' B '. We need to find $\mathrm{P}(\mathrm{A}$ or B$)$.

$\mathrm{P}(\mathrm{A}) = 3/6$ (odd numbers $= 1, 3$ and 5)

$P(B) = 4/6$ (numbers less than $5 = 1, 2, 3$ and 4)

$\mathrm{P}(\mathrm{A}$ and B$) = 2/6$ (numbers that are both odd and less than $5 = 1$ and 3)

Now, $P(A$ or $B) \quad = P(A) + P(B) - P(A$ or $B)$

$$= 3/6 + 4/6 - 2/6$$

$$= 5/6$$

***Example* 1.2.10:** A box contains 4 chocobar's and 4 ice creams. Tom eats 3 of them one after another. What is the probability of sequentially choosing 2 chocobar's and 1 ice-cream?

Solution:

Probability of choosing 1 chocobar $= 4/8 = 1/2$

After taking out 1 chocobar, the total number is 7 .

Probability of choosing 2 nd chocobar $= 3/7$

Probability of choosing 1 ice-cream out of a total of $6 = 4/6 = 2/3$

So the final probability of choosing 2 choc-bar's and 1 ice-cream $= 1/2 * 3/7 * 2/3 = 1/7$

***Example* 1.2.11:** When two dice are rolled, find the probability of getting a greater number on the first die than the one on the second, given that the sum should equal 8 .

Solution:

Let the event of getting a greater number on the first die be G. There are 5 ways to get a sum of 8 when two dice are rolled $= \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$.

And there are two ways where the number on the first die is greater than the one on the second given that the sum should equal $8, \mathrm{G} = \{(5,3),(6,2)\}$.

Therefore, P (Sum equals 8) $= 5/36$ and $P(G) = 2/36$.

Now,
$$\begin{aligned} P(G \mid \text{ sum equals } 8) &= P(G \text{ and sum equals } 8)/P(\text{ sum equals } 8) \\ &= (2/36)/(5/36) \\ &= 2/5. \end{aligned}$$

1.3 Random Variables

We have been examining with random experiments that can be described by finite sample spaces. We studied the assignment and computation of probabilities of events. In practice, one observes a function defined on the space of outcomes. Till now, we were concerned with such functions without defining the term *random variable*. Here we study the notion of a random variable and examine some of its properties.

The fundamental difference between a random variable and a real-valued function of a real variable is the associated notion of probability distribution. Nevertheless, our knowledge of advanced calculus or real analysis is the basic tool in the study of random variables and their probability distributions.

Definition 1.3.1. Let (Ω, S) be a sample space. A finite, single-valued function that maps Ω into $\mathbb{R}$ is called a *random variable*(R.V) if the inverse images under X of all Borel sets in $\mathbb{R}$ are events, that is, if

$$X^{-1}(B) = \{\omega : X(\omega) \in B\} \in S \quad \text{for all Borel sets } B. \tag{1.1}$$

Theorem 1.3.1. X is an R.V if and only if for each $x \in \mathbb{R}$,

$$\{\omega : X(\omega) \leq x\} = \{X \leq x\} \in S. \tag{1.2}$$

***Note* 2:** The notion of probability does not enter into the definition of an R.V.

Remark 1.3.1. If X is an R.V, the sets $\{X = x\}$, $\{a < X \leq b\}$, $\{X < x\}$, $\{a \leq X < b\}$, $\{a < X < b\}$, $\{a \leq X \leq b\}$ are all events. Moreover, we could have used any of these intervals to define an R.V.

Remark 1.3.2. One may ignore the technical conditions (1.1) and (1.2), and just remember random variables as mere real valued functions, but there exist real-valued functions defined on Ω that are not R.Vs but it is hard to encounter them in practical applications.

***Example* 1.3.1:** For any set $A \subseteq \Omega$, define

$$I_A(\omega) = \begin{cases} 0, & \omega \notin A, \\ 1, & \omega \in A. \end{cases}$$

$I_A(\omega)$ is called the *indicator function* of set A. I_A is an R.V if and only if $A \in S$.

***Example* 1.3.2:** Let $\Omega = \{H, T\}$, and S be the class of all subsets of Ω. Define X by $X(H) = 1$ and $X(T) = 0$. Then

$$X^{-1}(-\infty, x] = \begin{cases} \emptyset, & x < 0, \\ \{T\}, & 0 \leq x < 1, \\ \{H, T\}, & x \geq 1, \end{cases}$$

and we see that X is an R.V.

***Example* 1.3.3:** Let $\Omega = \{HH, TT, HT, TH\}$ and S be the class of all subsets of Ω. Define X by

$$X(\omega) = \text{ number of } H's \text{ in } \omega.$$

Then $X(HH) = 2$, $X(HT) = X(TH) = 1$ and $X(TT) = 0$.

$$X^{-1}(-\infty, x] = \begin{cases} \emptyset, & x < 0 \\ \{TT\}, & 0 \leq x < 1, \\ \{TT, HT, TH\}, & 1 \leq x < 2, \\ \Omega, & 2 \leq x. \end{cases}$$

Thus X is an R.V.

Remark 1.3.3. Let (Ω, S) be a discrete sample space; that is, let Ω be a countable set of points and S be the class of all subsets of Ω. Then every numerical-valued function defined on (Ω, S) is an R.V.

***Example* 1.3.4:** Let $\Omega = [0,1]$ and $S = \mathscr{B} \cap [0,1]$ be the σ-field of Borel sets on $[0,1]$. Define X on Ω by

$$X(\omega) = \omega, \quad \omega \in [0,1].$$

Clearly, X is an R.V. Any Borel set of Ω is an event.

1.3.1 Types of Random Variables

A random variable can be either discrete or continuous. Discrete random variables take on a countable number of distinct values. Consider an experiment where a coin is tossed three times. If X represents the number of times that the coin comes up heads, then X is a discrete random variable that can only have the values $0, 1, 2, 3$ (from no heads in three successive coin tosses to all heads). No other value is possible for X.

Continuous random variables can represent any value within a specified range or interval and can take on an infinite number of possible values. An example of a continuous random variable would be an experiment that involves measuring the amount of rainfall in a city over a year or the average height of a random group of 25 people.

Drawing on the latter, if Y represents the random variable for the average height of a random group of 25 people, you will find that the resulting outcome is a continuous figure since height may be 5 ft or 5.01 ft or 5.0001 ft . Clearly, there is an infinite number of possible values for height.

A random variable has a probability distribution that represents the likelihood that any of the possible values would occur. Let's say that the random variable, Z, is the number on the top face of a die when it is rolled once. The possible values for Z will thus be $1, 2, 3, 4, 5,$ and 6. The probability of each of these values is $1/6$ as they are all equally likely to be the value of Z.

For instance, the probability of getting a 3, or $\mathrm{P(Z = 3)}$, when a die is thrown is $1/6$, and so is the probability of having a 4 or a 2 or any other number on all six faces of a die. Note that the sum of all probabilities is 1.

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TET (Teachers Eligibility Test)

TET **தேர்விற்கு நம்** ***Teachers Care Academy*****-யில்** ***Paper I*** **மற்றும்** ***Paper II*** **என இரண்டு தாள்களுக்கும் பிரத்தியேகமாக பயிற்சிகளை வழங்குகிறோம்**

இதற்கு தமிழ்நாடு அரசால் வழங்கப்பட்டுள்ள பள்ளி பாட புத்தகத்தில் இருந்து குறிப்புகளை எடுத்து ***Study Material*****-களாக வழங்குகிறோம்**

மேலும் ***Psychology*****-க்கு** ***TRB*****-ஆல்வழங்கப்பட்டுள்ள பாடத்திட்டத்தை பின்பற்றி பல்வேறு** ***Reference Book*****-லிருந்து குறிப்புகளை எடுத்து** ***Study Material*****-களாக வழங்குகிறோம்**

UGTRB

TET **தேர்வில் வெற்றி பெற்ற ஆசிரியர்களுக்கு நடத்தப்படும்** ***UGTRB*** **போட்டி தேர்வுக்காகஅனைத்து மொழி பாடத்திற்கும் பயிற்சிகளை வழங்கி வருகிறோம் அதாவது**

- *Tamil*
- *English*
- *Mathematics*
- *Physics*
- *Chemistry*
- *Botany*
- *Zoology*
- *History*
- *Geography*

TCA

SGTRB

TET **தேர்வில் வெற்றி பெற்ற ஆசிரியர்களுக்கு நடத்தப்படும்** ***SGTRB*** **போட்டி தேர்வுக்காக தமிழ்நாடு அரசால் வழங்கப்பட்டுள்ள பள்ளி பாட புத்தகத்தில் இருந்து குறிப்புகளை எடுத்து** ***Study Material*****-களாக வழங்குகிறோம்**

BEO

BEO **தேர்வுக்காக** ***TRB*****-ஆல் பாடத்திட்டம் வெளியிடப்பட்டுள்ளது அந்த பாடத்திட்டத்தின் அடிப்படையில் அனைத்து பாடத்திற்கும் உங்கள்** ***Teachers Care Academy*** **அலகு** ***(Unit-Wise)*** **வாரியாக** ***Study Material*****-களை வழங்குகிறது.**

POLYTECHNIC TRB

Polytechnic **தேர்விற்காக உங்கள்** ***Teachers Care Academy*** **பின்வரும் மொழி பாடத்திற்கு பயிற்சிகளை வழங்கி வருகிறது. அதாவது,**

- *CIVIL*
- *EEE*
- *ECE*
- *CSE*
- *Mechanical*
- *English*
- *Mathematics*
- *Physics*
- *Chemistry*

College TRB

தமிழ்நாட்டில் அரசு கல்லூரிகளில் காலியாக உள்ள உதவி பேராசிரியர் பணிக்கு *TRB* வெகு விரைவில் போட்டித் தேர்வை நடத்த இருக்கிறது

அந்த தேர்வுக்காக நம் *Teachers Care Academy*-யில் பின்வரும் மொழி பாடத் திட்டத்திற்கும் பயிற்சிகளை வழங்கி வருகிறது

- *Tamil*
- *English*
- *Mathematics*
- *Physics*
- *Chemistry*
- *Botany*
- *Zoology*
- *Economics*
- *Commerce*
- *Computer Science*
- *History*
- *Geography*

TCA

Special Teachers

***TRB*-ஆல்நடத்தப்படும் சிறப்பாசிரியர் தேர்வுக்காக நம் *Teachers Care Academy*-யில் பின்வரும் பாடத்திட்டத்திற்கு பிரத்தியேகமாக பயிற்சிகள் வழங்கப்பட்டு வருகிறது. அதாவது,**

- *Sewing*
- *Drawing*
- *Music*
- *PET*

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Question, Answer with Explanation.

மேற்கண்ட அனைத்து தேர்வுகளுக்கும் உங்கள் *Teachers Care Academy* பலவிதமான பயிற்சிகளை வழங்குகிறது, அவை

இந்த ஆண்டு (2024) *TNPSC Batch*-யும் அறிமுகம் செய்திருப்பதில் நாங்கள் மகிழ்ச்சி அடைகிறோம். எங்கள் வழிகாட்டுதல் வரவிருக்கும் தேர்வுகளில் நீங்கள் வெற்றி பெற உதவும் என நாங்கள் உறுதியாக நம்புகிறோம்.

உங்கள் அரசு ஆசிரியர் பணி கனவு நிறைவேற வாழ்த்துக்கள்!

அன்புடன்,
Teacher's Care Academy